POSITIVE LINEAR FUNCTIONALS
WITHOUT REPRESENTING MEASURES

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Abstract. For $k$ even, let $\mathcal{P}_k$ denote the vector space of polynomials in 2 real variables of degree at most $k$. A linear functional $L : \mathcal{P}_k \rightarrow \mathbb{R}$ is positive if $p \in \mathcal{P}_k$, $p|x|^2 \geq 0 \implies L(p) \geq 0$. Hilbert’s theorem on sums of squares (cf. [15]) implies that $L : \mathcal{P}_4 \rightarrow \mathbb{R}$ is positive if and only if the moment matrix associated to $L$ is positive semidefinite. In this note, using $k = 6$, we exhibit the first family of positive linear functionals $L : \mathcal{P}_k \rightarrow \mathbb{R}$ whose positivity cannot be derived from the positive semidefiniteness of the associated moment matrices, and which do not correspond to integration with respect to positive measures.

1. Introduction

For $n \geq 1$, $k \geq 0$, let $\beta \equiv \beta^{(k)} := \{ \beta_i : i \in \mathbb{Z}_{+}^n, |i| \leq k \}$ denote an $n$-dimensional real multisequence of degree $k$ (where $i \equiv (i_1, \ldots, i_n)$ and $|i| = i_1 + \cdots + i_n$). Let $K \subseteq \mathbb{R}^n$ be a nonempty closed set. The truncated $K$-moment problem asks for conditions on $\beta$ so that there exists a positive Borel measure $\mu$ on $\mathbb{R}^n$ satisfying

$$\text{supp } \mu \subseteq K$$

and

$$\beta_i = \int_K x^i d\mu \quad (|i| \leq k)$$

(where $x \equiv (x_1, \ldots, x_n)$ and $x^i = x_1^{i_1} \cdots x_n^{i_n}$); we refer to such a measure as a $K$-representing measure for $\beta$.

Let $\mathcal{P} \equiv \mathbb{R}[x_1, \ldots, x_n]$ and let $\mathcal{P}_k$ denote the subspace consisting of all polynomials $p$ with $\text{deg } p \leq k$. We associate to $\beta$ the Riesz functional $L_{\beta} : \mathcal{P}_k \rightarrow \mathbb{R}$ defined by $L_{\beta}(\sum a_i x^i) = \sum a_i \beta_i$. Note that if $\beta$ admits a $K$-representing measure $\mu$, then $L_{\beta}$ is $K$-positive in the sense that $p \in \mathcal{P}_k$, $p|K \geq 0 \implies L_{\beta}(p) \geq 0$; indeed, in this case, $L_{\beta}(p) = \int_K p d\mu \geq 0$ (since $\mu$ is a positive measure supported in $K$). For $K = \mathbb{R}^n$, we refer to $\mu$ simply as a representing measure and to a $K$-positive functional $L_{\beta}$ as positive. For $K$ compact, the proof of Tchakaloff’s Theorem [20] shows that if $L_{\beta}$ is $K$-positive, then $\beta$ admits a $K$-representing measure. Even for $n = 1$, this implication does not always hold for non-compact $K$ [9, Example 2.1], but the following result of [9] characterizes the existence of $K$-representing measures in terms of $K$-positivity.

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THEOREM 1.1. ([9, Theorem 1.2]) Let $k = 2m$ or $k = 2m + 1$. $β ≡ β^{(k)}$ admits a $K$-representing measure if and only if $L_β$ admits a $K$-positive extension $L_β : \mathcal{P}_{2m+2} → \mathbb{R}$.

Theorem 1.1 is the analogue for the truncated moment problem of the classical theorem of M. Riesz ($n = 1$) [17] and E.K. Haviland ($n > 1$) [13], which shows that a full multisequence $β ≡ β^{(ω)}$ admits a $K$-representing measure if and only if the corresponding Riesz functional $L_β : \mathcal{P} → \mathbb{R}$ is $K$-positive. (Theorem 1.1 actually implies the Riesz-Haviland Theorem (see [9]).) In view of Theorem 1.1 and the Riesz-Haviland Theorem, it is important to be able to recognize when a functional $L_β$ is $K$-positive. A celebrated theorem of K. Schmüdgen [19] shows that if $K$ is a compact semialgebraic set, then a strictly positive polynomial on $K$ can be expressed as a weighted sum of squares, and this permits one to establish $K$-positivity for $L_β^{(ω)}$ simply by verifying the positive semidefiniteness of a finite number of localizing matrices associated to $β^{(ω)}$ (cf. [9] [14] [19]). Nevertheless, a basic difficulty is that for a general closed (even semialgebraic) set $K$, there is no concrete description of the polynomials that are nonnegative on $K$, so there may be no direct test for $K$-positivity of $L_β$ or $L_β^{(ω)}$.

[12, Theorem 2.2] shows that $L_β^{(k)}$ is $K$-positive if and only if $β^{(k)}$ is in the closure of the multisequences having $K$-representing measures. Motivated by this result, in the sequel we will use an approximation technique to provide the first examples of $K$-positive functionals $L_β^{(k)}$ in cases where $β^{(k)}$ has no $K$-representing measure and where the strictly positive polynomials on $K$ cannot be represented as weighted sums of squares.

To explain our results, let us first recall the scope of the sums of squares approach when $K = \mathbb{R}^n$. Following [7], for $k = 2d$, we may associate to $β ≡ β^{(k)}$ the moment matrix $M ≡ M_d(β)$, of size $\text{dim } \mathcal{P}_d$, defined by

$$\langle M\hat{p}, \hat{q} \rangle = L_β(p\hat{q}) \quad (p, q ∈ \mathcal{P}_d)$$  \hspace{1cm} (1.1)

(where $\hat{p}$ is the vector of coefficients of $p$ relative to the basis of monomials of $\mathcal{P}_d$ in degree-lexicographic order). If $β$ admits a representing measure $μ$, then for $q ∈ \mathcal{P}_d$,

$$\langle M\hat{q}, \hat{q} \rangle = L_β(q^2) = ∫ q^2 dμ ≥ 0,$$

so $M$ is positive semidefinite ($M ≥ 0$). In general, even for $n = 1$, positive semidefiniteness of $M$ is not sufficient for $β$ to admit a representing measure [3]. However, if $M ≥ 0$ and each polynomial $p$ in $\mathcal{P}_{2d}$ that is strictly positive on $\mathbb{R}^n$ admits a sum of squares decomposition, i.e., $p = \sum q_j^2 (q_j ∈ \mathcal{P}_d)$, then $L_β$ is positive. Indeed, in this case,

$$L_β(p) = \sum L_β(q_j^2) = \sum \langle M\hat{q}_j, \hat{q}_j \rangle ≥ 0;$$  \hspace{1cm} (1.2)

now, if $q∥\mathbb{R}^2 ≥ 0$, then for every $ε > 0$, $q + ε$ is strictly positive, so (1.2) implies $L_β(q) ≥ εL_β(1)$, and thus $L_β(q) ≥ 0$.

A well-known theorem of Hilbert (cf. [15] [16]) shows that for $k$ even, every polynomial in $\mathcal{P}_k$ that is nonnegative on $\mathbb{R}^n$ may be expressed as a sum of squares of polynomials if and only if $n = 1$; or $n = 2$ and $k = 4$; or $n ≥ 1$ and $k = 2$. We note that these are precisely the cases in which each polynomial that is strictly positive on $\mathbb{R}^n$ is a sum of squares, so that precisely in these cases can positivity of $L_β$ be established.
through the positive semidefiniteness of $M_k(\beta)$ (as in the argument following (1.2)). To see this, suppose $q \in \mathcal{P}_k$ is nonnegative on $\mathbb{R}^n$ but is not a sum of squares. It suffices to show that for all sufficiently small $\varepsilon > 0$, the strictly positive polynomial $q + \varepsilon$ is not a sum of squares. Let $\Sigma_k$ denote the convex cone in $\mathcal{P}_k$ consisting of sums of squares and recall from [14, Section 3.8] that $\Sigma_k$ is closed in $\mathcal{P}_k$ (relative to the norm $||\Sigma_k x^T|| = \max |a_i|$). It follows from the Minkowski separation theorem [2, (34.2)] that there exists a linear functional $L : \mathcal{P}_k \rightarrow \mathbb{R}$ for which $L|\Sigma_k \geq 0$ and $L(q) < 0$. By continuity, $L(q + \varepsilon) < 0$ for all sufficiently small $\varepsilon > 0$, so $q + \varepsilon$ is not a sum of squares.

Let us apply the preceding observations when $n = 2$. For $n = 2$ and $k = 4$, Hilbert’s theorem and (1.2) imply that if $\beta \equiv \beta^{(4)}$ satisfies $M_2(\beta) \succeq 0$, then $L_\beta$ is positive. However, the above discussion implies that there exists $\beta \equiv \beta^{(6)}$ such that $M_3(\beta)$ is positive semidefinite but $L_\beta$ is not positive; moreover, an example of Schmüdgen [18] illustrates a case where $M_3(\beta)$ is positive definite but $L_\beta$ is not positive. In view of these examples, for $\beta \equiv \beta^{(6)}$ we cannot use sums of squares, as in (1.2), to promote positive semidefiniteness of $M_3(\beta)$ into positivity of $L_\beta$. In cases where $\beta^{(6)}$ fails to have a representing measure, we require a new technique, beyond sums of squares, to establish that $L_\beta$ is positive, and the goal of this note is to illustrate such a technique.

In the sequel, for $n = 2$, we denote the successive rows and columns of the moment matrix $M \equiv M_3(\beta)$ by $1, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3$. We denote the elements of $\beta^{(6)}$ by $\beta_{ij}$ ($i, j \geq 0$, $i + j \leq 6$), where $\beta_{ij}$ corresponds to the monomial $x^iy^j$. Let Col $M$ denote the column space of $M$ in $\mathbb{R}^{10}$. Under the conditions

$$M \equiv M_3(\beta) \succeq 0, \ Y = X^3 \text{ in Col } M, \ \text{rank}(M) = 9,$$  \hspace{1cm} (1.3)

we will associate to $\beta$ an expression $\psi(\beta)$, a certain rational function of the moment data in $\beta$ (see Section 2). [11, Theorem 1.1] implies that under the conditions of (1.3), $\beta$ has a representing measure (necessarily supported in the curve $y = x^3$) if and only if $\beta_{15} > \psi(\beta)$. Our main result, which follows, displays a family of positive functionals $L_\beta^{(6)}$ whose positivity does not arise from the existence of representing measures or from sums of squares as in (1.2).

**Theorem 1.2.** For $\beta \equiv \beta^{(6)}$, suppose $M \equiv M_3(\beta) \succeq 0, \ Y = X^3$, and rank $M = 9$. If $\beta_{15} = \psi(\beta)$, then $\beta$ has no representing measure, but $L_\beta$ is positive.

**Remark 1.3.** i) The positivity of $L_\beta$ was conjectured in [12, Example 2.5], where it was established for particular numerical instances of $M$ satisfying (1.3); the key new ingredient for the proof of Theorem 1.2 is Proposition 2.2, which is proved by means of a highly intensive symbolic algebra calculation. The functionals of Theorem 1.2 (including those of [12, Example 2.5]) seem to be the first concrete examples of $L_\beta$ where positivity cannot be established through the existence of representing measures or via sums of squares.

ii) We may view a multisequence $\beta \equiv \beta^{(k)}$ as an element of $\mathbb{R}^\eta$, where $\eta = \dim \mathcal{P}_k$. Let $\mathcal{C}$ denote the convex cone in $\mathbb{R}^\eta$ consisting of those multisequences $\beta$ having $K$-representing measures. [12, Theorem 2.2] shows that $L_\beta$ is $K$-positive if
and only if $\beta \in \overline{C}$. From this viewpoint, with $k = 6$ and $K = \mathbb{R}^2$, the sequences $\beta$ in Theorem 1.2 belong to $\text{bdry } C \setminus C$.

The case of the bivariate moment problem for $\beta^{(4)}$ with $M_2(\beta^{(4)})$ singular was solved in [6] [8]: concrete necessary and sufficient conditions for representing measures are known, and finitely atomic representing measures with the fewest atoms can be explicitly computed. Note that in Theorem 1.2, J. Nie and the second-named author proved that for a bivariate $\beta \equiv \beta^{(4)}$, if $M_2(\beta) \succeq 0$, then $\beta$ does have a representing measure, but at present it is not known how to construct such a measure. It follows from [1, Theorem 2] that since there is a representing measure, then there exists a cubature rule $\nu$ (a finitely atomic representing measure) with $\text{card supp } \nu \leq \dim P_4 = 15$, but there is no method known for computing $\nu$. Further, any representing measure $\mu$ necessarily satisfies $\text{card supp } \mu \succeq \text{rank } M_2(\beta) = 6$ (cf. [4] [7]), but it remains unknown whether 6-atomic representing measures always exist. In general, if a positive semidefinite $M_d$ admits a flat (i.e., rank-preserving) extension $M_{d+1}$, then $\beta^{(2d)}$ admits a rank $M_d$-atomic representing measure that can be explicitly computed [7], but it is unknown whether a positive definite $M_2$ always admits a flat extension $M_3$ (and a corresponding 6-atomic measure). We will show that for the sequences $\beta^{(4)}$ which appear in Theorem 1.2, there always exist computable 9-atomic representing measures:

**Corollary 1.4.** If $\beta^{(4)}$ admits an extension to a sequence $\beta^{(6)}$ satisfying (1.3) and $\beta_{15} \geq \psi(\beta^{(6)})$, then $\beta^{(5)}$ admits a computable 9-atomic representing measure.

### 2. A family of positive functionals $L_{\beta^{(6)}}$

In this section we prove Theorem 1.2, and to this end we require a preliminary result and some notation. Recall that a real symmetric $2 \times 2$ block matrix $M \equiv \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ is positive semidefinite if and only if $A \succeq 0$, $B = AW$ for some matrix $W$ (equivalently, $\text{Ran } B \subseteq \text{Ran } A$), and $C \succeq W^TAW$ (cf. [7]). For a bivariate moment matrix $M \equiv M_d(\beta)$, we denote the columns by $1$, $X$, $Y$, $X^0$, $X_d$, $X_{d-1}Y$, $\ldots$, $XY^{d-1}$, $X^d$ (following the degree-lexicographic ordering of the monomials in $\mathcal{P}_d$). Each linear dependence relation in the columns of $M$ may thus be expressed as $p(X,Y) = 0$, where $p(x,y) \equiv \sum a_{ij}x^iy^j \in \mathcal{P}_d$ and $p(X,Y) := \sum a_{ij}X^iY^j$. Recall from [7] that $M$ is recursively generated if $p$, $q$, $pq \in \mathcal{P}_d$, $p(X,Y) = 0 \implies (pq)(X,Y) = 0$. Recursiveness is a necessary condition for the existence of a representing measure for $\beta$. The following result is implicit in [11].

**Proposition 2.1.** If $\beta \equiv \beta^{(2d)}$ has a representing measure, then $M \equiv M_d(\beta)$ admits a positive, recursively generated moment matrix extension

\[
M_{d+1}(\bar{\beta}) \equiv \begin{pmatrix} M & B(d+1) \\ B(d+1)^T & C(d+1) \end{pmatrix};
\]

in particular, $\text{Ran } B(d+1) \subseteq \text{Ran } M$. 

Proof. Since $\beta$ has a representing measure, say $\mu$, it follows from [1] that $\mu$ admits a cubature rule of degree $2d$, i.e., there exists a finitely atomic positive measure $\nu$ that has the same moments as $\mu$ up to (at least) degree $2d$. Since $\nu$ is finitely atomic, it has finite moments of degrees $2d+1$ and $2d+2$, and a corresponding moment sequence $\tilde{\beta} \equiv \tilde{\beta}^{(2d+2)}$. Since $\nu$ is obviously a representing measure for $\tilde{\beta}$, it follows that $M_{d+1}(\tilde{\beta})$ is a positive, recursively generated extension of $M$, and, in particular, $\text{Ran } B(d+1) \subseteq \text{Ran } M$. □

We note that the preceding result holds for general $n \geq 1$.

We next describe how to construct $M_3(\beta)$ satisfying the hypotheses of Theorem 1.2. Let $n = 2$ and $d = 3$. We consider the general form of a moment matrix $M_3(\beta)$ with a column relation $Y = X^3$ (normalized with $\beta_{00} = 1$):

$$M \equiv M_3(\beta) = \begin{pmatrix}
1 & a & b & c & e & d & b & f & g & x \\
a & c & e & b & f & g & e & d & h & j \\
b & e & d & f & g & x & d & h & j & k \\
c & b & f & e & d & h & f & g & x & u \\
e & f & g & d & h & j & g & x & u & v \\
d & g & x & h & j & k & x & u & v & w \\
b & e & d & f & g & x & d & h & j & k \\
f & d & h & g & x & u & h & j & k & r \\
g & h & j & x & u & v & j & k & r & s \\
x & j & k & u & v & w & k & r & s & t
\end{pmatrix}.$$ (2.1)

For suitable values of the moment data, $M$ satisfies the following properties:

$$M \succeq 0, \quad Y = X^3, \quad \text{rank } M = 9;$$ (2.2)

this is the case, for example, with

$$a = b = f = g = u = v = w = x = 0, \quad c = 1, \quad e = 2, \quad d = 5, \quad h = 14, \quad j = 42, \quad k = 132, \quad r = 429, \quad s = 1429, \quad t = 4847,$$ (2.3)

$$M \equiv M_3(\beta) = \begin{pmatrix}
1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\
0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\
1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\
0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\
0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 429 \\
0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 429 & 1429 \\
0 & 42 & 132 & 0 & 0 & 0 & 132 & 429 & 1429 & 4847
\end{pmatrix}.$$ (2.4)

In [11] we solved the truncated $K$-moment problem for $K = \{(x, y) \in \mathbb{R}^2 : y = x^3\}$. In particular, [11] provides a numerical test, that we next describe, for the existence of $K$-representing measures whenever $M$ as in (2.1) satisfies (2.2). From Proposition 2.1,
we know that if $\beta$ admits a representing measure, then $M$ admits a positive, recursively generated extension $M_4(\tilde{\beta})$. In any such extension, the moments must be consistent with the relation $y = x^3$, so in particular, we must have $\beta_{44} = \beta_{15}(\equiv s)$. To insure positivity, $M_4(\tilde{\beta})$ must satisfy a lower bound for the diagonal element $\beta_{44}$ (in row $X^2Y^2$, column $X^2Y^2$), which we may derive as in [11]. Let $J$ denote the compression of $M$ obtained by deleting row $X^3$ and column $X^3$; thus, $J \succ 0$. Let us write

$$J = \begin{bmatrix} N & U \\ U^T & \Delta \end{bmatrix},$$

where $N$ is the compression of $J$ to its first 8 rows and columns, $U$ is a column vector, and $\Delta \equiv \beta_{06}(\equiv t) > 0$. Consider the corresponding block decomposition of $J^{-1}$, which is of the form

$$J^{-1} = \begin{bmatrix} P & V \\ V^T & \varepsilon \end{bmatrix},$$

where $P \succ 0$ and $\varepsilon > 0$. Since $M_4(\tilde{\beta})$ is recursively generated, the relation $Y = X^3$ in $\text{Col } M$ propagates to the column relations $X^4 = XY$ and $X^3Y = Y^2$ in $\text{Col } M_4(\tilde{\beta})$, so by moment matrix structure, after deleting the element in row $X^3$, the first 8 remaining elements of column $X^2Y^2$ must be $W \equiv (h, x, u, j, k, r, v, w)^T$. Let $\omega = \langle PW, W \rangle$ and define

$$\psi(\beta) := \frac{\omega \varepsilon - \langle V, W \rangle^2}{\varepsilon}. \quad (2.5)$$

In [11] we showed that in $M_4(\tilde{\beta})$ we must have $\beta_{44} \geq \psi(\beta)$, so in $M$ we require $\beta_{15} \geq \psi(\beta)$, and [11, Theorem 1.1] implies that $\beta$ has a representing measure if and only if $\beta_{15} > \psi(\beta)$; since $Y = X^3$ in $\text{Col } M$, such a measure is necessarily supported in $K$ (cf. [7]).

Although $\psi(\beta)$ is formally defined in terms of all of the moments in $\beta$, we will show below (Proposition 2.2) that for $M$ as in (2.1)-(2.2), the value of $\psi(\beta)$ is actually independent of $s$ and $t$. For any $M$ satisfying (2.1) and (2.2), so that $\psi(\beta)$ is independent of $s$ and $t$, we now specify $s = \beta_{15} = \psi(\beta)$, and we adjust $t$ (if necessary) so that $M$ continues to be positive with $\text{rank } M = 9$. Thus, $M$ satisfies all of the hypotheses of Theorem 1.2; for a numerical example, consider (2.3), where a calculation shows that $\beta_{15} = 1429 = \psi(\beta)$.

**Proof of Theorem 1.2.** With $\beta$ as in the hypothesis, we claim that $L_\beta$ is positive. Since $\beta_{15} = \psi(\beta)$, positivity for $L_\beta$ cannot be derived from the existence of a representing measure, since [11, Theorem 1.1] shows that $\beta$ has no representing measure. Moreover, as we discussed in Section 1, positivity for $L_\beta$ cannot be derived from the positivity of $M$ via sums of squares arguments as in (1.2) because, by Hilbert’s theorem, there exist degree 6 bivariate polynomials that are everywhere nonnegative but are not sums of squares. To prove that $L_\beta$ is positive, we employ a sequence of approximate representing measures. Since $J \succ 0$, then $t \equiv \Delta > U^T N^{-1} U$. Thus, there exists $\delta > 0$ such that if we replace $s (= \psi(\beta))$ by $s + \frac{1}{m}$ (with $\frac{1}{m} < \delta$), then the resulting moment matrix, $M_3(\tilde{\beta}^{[m]})$, remains positive, with $\text{rank } M_3(\tilde{\beta}^{[m]}) = 9$ and $Y = X^3$ in
Col $M_3(\beta^{[m]})$. Since, from Proposition 2.2 (below), the value of $\psi(\beta^{[m]})$ is independent of $\beta_{15}[\beta^{[m]}]$ and $\beta_{06}[\beta^{[m]}]$, we have $\psi(\beta^{[m]}) = \psi(\beta) = s < \frac{1}{m} = \beta_{15}[\beta^{[m]}]$. It now follows from [11, Theorem 1.1] that $\beta^{[m]}$ has a representing measure, whence $L_{\beta^{[m]}}$ is positive. Note that the convex cone $\{ \beta \equiv \beta^{(6)} \in \mathbb{R}^{10} : L_\beta \text{ is positive} \}$ is closed; since $\| \beta^{[m]} - \beta \| = \frac{1}{m} \rightarrow 0$, we conclude that $L_\beta$ is positive. \hfill \box

To complete the proof of Theorem 1.2 it now suffices to prove the following result.

**Proposition 2.2.** For $M \equiv M_3(\beta)$ as in (2.1)-(2.2), the value of $\psi \equiv \psi(\beta)$ is independent of $s$ and $t$, i.e., if $M_3(\beta')$ is as in (2.1)-(2.2) and $\beta'$ agrees with $\beta$ except possibly in the values of $s$ and $t$, then $\psi(\beta') = \psi(\beta)$.

**Proof.** Our proof that the value of $\psi$ is independent of $s$ and $t$ is computational. We represent $J^{-1}$ in the form

$$J^{-1} = \begin{bmatrix} P & V \\ V^T & \varepsilon \end{bmatrix} = \frac{1}{D} \begin{bmatrix} P' & V' \\ V'^T & \varepsilon' \end{bmatrix},$$

where $D = \det J$. Then we have

$$\psi = \frac{\langle P'W, W \rangle - \frac{1}{D} \langle V', W \rangle^2}{D}. \quad (2.6)$$

One can verify, either computationally or by examining the relevant terms, that:

i) $\langle P'W, W \rangle$ is a polynomial in $s$ and $t$ of the form $A_1 + A_2s + A_3s^2 + A_4t$, with coefficients $A_i$ that are polynomials in the remaining moment variables that define $M$.

ii) $\langle V'^T, W \rangle$ is a linear polynomial in $s$, with coefficients that are polynomials in the remaining moment variables, excluding $t$.

iii) $D$ is quadratic in $s$ and linear in $t$ (as in i).

iv) $\varepsilon'$ is a polynomial in the moment variables, but with no terms including $s$ or $t$.

$\psi$ can then be represented as ratio of two polynomials in $s$ and $t$:

$$\psi = \frac{n_0 + n_1s + n_2s^2 + n_3t}{d_0 + d_1s + d_2s^2 + d_3t}, \quad (2.7)$$

where $\{n_i\}_{i=0}^3$ and $\{d_i\}_{i=0}^3$ are sequences of polynomial functions in the variables $a, b, f, g, u, v, w, c, e, d, h, j, k, r,$ and $x$ (and which contain no terms with $s$ or $t$).

Then the value of $\psi$ is independent of $s$ and $t$ if and only if

$$\frac{n_0}{d_0} = \frac{n_1}{d_1} = \frac{n_2}{d_2} = \frac{n_3}{d_3}. \quad (2.8)$$

Our proof of (2.8) consists of computing $f_0 = \frac{n_0}{d_0}, f_1 = \frac{n_1}{d_1}, f_2 = \frac{n_2}{d_2}, f_3 = \frac{n_3}{d_3}$ and showing that

$$\frac{f_0}{f_3} = \frac{f_1}{f_3} = \frac{f_2}{f_3} = 1. \quad (2.9)$$
By explicit computation using the computer algebra system Maxima, we were able to verify (2.9), thus completing the proof. However, the calculations of the terms involved in verifying (2.9) present particular computational challenges that we discuss in the next section. □

Proof of Corollary 1.4. Suppose $M_2 \succ 0$ and that $M_2$ can be extended to $M_3 (\beta)$ satisfying (1.3) and $\beta_{15} \geq \psi (\beta)$. If $\beta_{15} > \psi (\beta)$, then [11, Theorem 1.1] implies that $M_3$ admits a flat extension $M_4$, so $\beta^{(6)}$ admits a 9-atomic representing measure that can be explicitly constructed using the method of [7]. If $\beta_{15} = \psi (\beta)$, then Theorem 1.2 implies that $L_{\beta^{(6)}}$ is positive, so Theorem 1.1 implies that $\beta^{(5)}$ has a representing measure. For a more constructive approach, we recall that in the proof of Theorem 1.2., $\beta^{[m]}$ satisfies the conditions of [11, Theorem 1.1], which implies that $\beta^{[m]}$ has a 9-atomic representing measure that can be explicitly constructed using the method of [7]; clearly, such a measure is also a representing measure for $\beta^{(5)}$. □

We conclude with a question.

**Question 2.3.** If $M_3 (\beta)$ satisfies (2.2) and $\beta_{15} < \psi (\beta)$, can $L_\beta$ be positive?

If, in Question 2.3, $L_\beta$ were positive, then, from [12, Theorem 2.2], $\beta = \lim \beta^{[n]}$, where each $\beta^{[n]}$ has a representing measure. Such $\beta^{[n]}$ cannot all satisfy (2.2), for otherwise $\beta_{15} > \psi (\beta^{[n]})$ from [11, Theorem 1.1], implying $\beta_{15} \geq \psi (\beta)$. However, there may be sequences having representing measures which approximate $\beta$ and which do not satisfy (2.2).

### 3. Appendix

In order to establish (2.9), we first need to compute $J^{-1}$, and to then represent $\psi$ in the form (2.6). We can then extract from (2.6) the coefficients of various powers of $s$ and $t$ in the numerator and denominator, as well as terms independent of $s$ and $t$, so as to calculate the quantities $\{n_i\}_{i=0}^3$ and $\{d_i\}_{i=0}^3$ in (2.7). This then allows us to compute $\{f_i\}_{i=0}^3$ for use in verifying (2.9).

We first attempted to perform these computations using Mathematica software. On a dual-core x86_64 computer with 10GB RAM (memory for computation) and 8GB swap space (auxiliary storage), and using the Linux operating system, the program ran for several days without producing even $J^{-1}$. We noted that as the computation progressed, available RAM and swap space became almost wholly consumed, eventually shutting down the operating system. We repeated the above calculations with Maple symbolic algebra system, but Maple also failed to yield results. We then tried the same computations using Matlab, which employs the Maple symbolic algebra system, but Matlab also failed to yield any result, eventually shutting down the operating system.

We then turned to Maxima, an open source computational software system descended from the now-extinct commercial Macsyma software. On the above-mentioned machine, Maxima was able to compute $J^{-1}$ and the representation of $\psi$ as in (2.7), and store the results on disk, in approximately 100 minutes. The intermediate expressions
leading up to each \( f_i \) \((0 \leq i \leq 3)\) are very large, taking several hundred megabytes of storage for each. Nevertheless, Maxima was able to retrieve the components of \( \psi \) from disk, and compute and simplify all of the expressions \( f_1, f_2 \) and \( f_3 \), in approximately 1 hour. Although Maxima also computed \( f_0 \), its simplification via the ratsimp function in Maxima failed, with Maxima reporting that the computation was aborted due to heap space exhaustion. Now Maxima is written in the Lisp programming language, and relies on a Lisp engine to do its computations. On our Linux operating system, Maxima was using the SBCL Lisp engine, Steel Box Common Lisp, another open source software, to perform the calculations. Because the source code was accessible to us, we recompiled SBCL so that it would be able to utilize up to 25GB of heap space. On this recompiled Lisp engine, Maxima loaded the components of \( \psi \) from disk and computed and simplified \( f_0 \) in approximately 4200 seconds of cpu time (and an amount of real time that varied, on several trials, between 2.2 hours and 8.5 hours, depending on overall system conditions). The final outcome of these calculations is that each of the quantities \( \{f_i\}_{i=0}^{3} \) in (2.9) reduces to the same expression (requiring 2,343,199 bytes of memory and about 50 pages to print) for arbitrary values of the moment variables (not including \( s \) and \( t \)). The entire computation described above thus establishes (2.9) and proves Proposition 2.2.

The details of the calculations that eventually proved successful, including annotated Maxima code, are posted online at http://cs.newpaltz.edu/~easwaran/PLF. It is not clear to us why Maxima was successful, as compared to other software. This seems partly due to the way the computations are organized. For example, the Mathematica command Inverse[\( J \)] is carried out with the enormous term Det[\( J \)] present in the denominator of each entry of the inverse, whereas Maxima provides an option for carrying \( 1/(\text{det} \ J) \) outside, thus reducing storage requirements on swap space. We suspect that the use of Lisp as its underlying processing mechanism also contributes to the success of Maxima. In particular, the open source nature of Maxima and of the SBCL Lisp processor allowed us to recompile source code, which was crucial to the success of the computation (as described above). (More information about Maxima and SBCL can be found at http://maxima.sourceforge.net/ and http://sbcl.sourceforge.net/.)

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