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The *g*-extra diagnosability of the balanced hypercube under the PMC and MM* model

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Abstract

Fault diagnosis plays an important role in the measuring of the fault tolerance of an interconnection network, which is of great value in the design and maintenance of large-scale multiprocessor systems. As a classical variant of the hypercube, the Balanced Hypercube, denoted by $BH_n(n \geq 1)$, has drawn a lot of research attention, and its g-extra diagnosability has been studied to improve the network diagnostic ability. However, the current literatures on g-extra diagnosability of BH_n under the PMC model only cover the cases of $g < 6$, and what's more, seldom involve its g-extra diagnosability under the MM* model, which is a great limitation on the research of BH_n diagnosability. In this paper, the upper and lower bounds of the g-extra diagnosability of the balanced hypercube are proved, respectively, based on the g-extra connectivity by the contradiction method, and finally, the g-extra diagnosability of BH_n for $2 \leq g \leq 2n-1$ under the PMC and MM* model is obtained, i.e., $2\left[(n-2)\lceil \frac{g-1}{2} \rceil + n\right] + g$. In addition, as a special case, the g-extra diagnosability of the balanced hypercube for $g = 2n$ is proved to be $2^{2n-1} - 1$ under the PMC and MM* model. In the end, simulation experiments are conducted to verify the effectiveness of our proposed theories. The conclusion of this paper has certain theory and application value for the research of BH_n fault diagnosis.

Keywords g-extra diagnosability · Balanced hypercube · Interconnection networks · PMC model · MM* model

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1 Introduction

Due to the rapid development of network technology and the expansion of network scale, there are more and more processors in large-scale multiprocessor systems, in which the existence of faulty nodes is often inevitable, and it is necessary to make in-depth explore on fault diagnosis features of interconnection networks.

Preparat et al. proposed the PMC model in 1967 [1], and it stipulates that any two processors that are neighbors in the system could test each other. After decades, Sengupta et al. put forward the MM* model [2], that any test node w in the system can test any two adjacent nodes. If the system G could diagnose all the faulty nodes at the same time, and the number of faulty nodes does not exceed t, the system G is called t-diagnosable [1]. It requires that the number of correct nodes must be greater than the number of failed nodes, which imposes certain restrictions on the diagnosis of node diagnosis [3]. In 2005, the conditional diagnosability was proposed by setting more restrictions [4]. The conditional diagnosis method requires that the neighbors of each node cannot fail at the same time, which covers most of the failure cases and greatly improved the diagnosability of interconnection networks. But the conditional diagnosis strategy performs not well enough with the increase of interconnection network dimensions [3], because it only assumes one non-faulty neighbor node, and the assumption is not so practical for high-dimensional networks. In 2012, based on R_g connectivity, the g-good-neighbor diagnosability was proposed [5], and it requires at least g faulty-free neighbor nodes for each non-faulty node. The g-good-neighbor diagnosability is more suitable for the high-dimensional interconnection networks than the conditional diagnosability. However, the g-good-neighbor diagnosability only imposes certain restrictions on the correct nodes, but not on the failed nodes [6]. In 2015, Zhang et al. proposed a new diagnostic method, called g-extra diagnosability, which is defined under the assumption that each connected component of $G-F$ has at least $(g+1)$ fault-free nodes, where F denotes fault node set and $F \subseteq V$ [7]. Deriving from the g-good-neighbor diagnosability, the g-extra diagnosability draws much research attention owning to its stronger fault diagnosability and higher accuracy of reliability measurement in heterogeneous environments. In literatures [7–12, 28], the g-extra diagnosability of some classic interconnection networks are explored under the PMC and MM* models, including the hypercube, the folded hypercube, the twisted cube, arrangement graphs, and the crossed cube. Some researchers also investigated the relationship between reliability and diagnostics, for example, Liu [13] et al. established the relationship between g-extra conditional diagnosability and g-extra connectivity of graphs under the MM* model.

In recent years, as an important variant of hypercubes, researches on topological properties of the balanced hypercube have been made, such as the structural fault tolerance of the balanced hypercube [14], the connectivity of the structure and substructures [15], the problem of two node-disjoint paths with $2n-3$ faulty vertices [16], g-extra connectivity [17], Hyper-Hamiltonian laceability [18], and conditional diagnosability under the MM* model [19]. Studies on the reliability

[20] and g-extra diagnosability of the balanced hypercube are made under the PMC model as well. In 2019, Lin et al. proved the g-extra diagnosability of BH_n when $1 \leq g \leq 3$ [21], and in 2019, Zhang et al. further proved the g-extra diagnosability for $4 \leq g \leq 5$ [22]. However, their conclusions applies only to $1 \leq g \leq 5$ and do not consider the cases of $6 \leq g$. Therefore, in this paper, we will mainly discuss the g-extra diagnosability of BH_n when $g \geq 6$ under the PMC model. At the same time, we find that there is even few research on the g-extra diagnosability of BH_n under the MM* model, so we will also prove the g-extra diagnosability when $g \geq 2$ under the MM* model.

In this paper, based on the g-extra connectivity [23] of BH_n, which is the minimum cardinality of g-extra cuts and is necessary to prove the g-extra diagnosability, we prove the g-extra diagnosability of BH_n (denoted by $t_g(BH_n)$) under the PMC and MM* model, respectively, and obtain a unified g-extra diagnosability formula, i.e., $2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$ for $2 \leq g \leq 2n-1$. In addition, we also prove that the g-extra diagnosability of BH_n is $2^{2n-1} - 1$ when $g = 2n$ under the PMC and MM* model.

The primary contributions of our work are as follows:

- We investigate the g-extra diagnosabilities of BH_n under PMC and MM* model.
- We prove that $t_g(BH_n) = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$ for $2 \leq g \leq 2n-1$, and $t_g(BH_n) = 2^{2n-1} - 1$ for $g = 2n$ under the PMC model. We extend the existing g-extra diagnosabilty of the balanced hypercube under the PMC model to a more general condition, i.e., $2 \leq g \leq 2n$.
- We prove that $t_g(BH_n) = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$ for $2 \leq g \leq 2n-1$, and $t_g(BH_n) = 2^{2n-1} - 1$ for $g = 2n$ under the MM* model, which is seldomly reported in current literatures.

A comparative analysis of the g-extra diagnosability in this paper and previous studies on the g-extra diagnosability of BH_n under the PMC and MM* model is shown in Table 1.

The rest of this paper is organized as follows. Section 2 introduces some basic knowledge of graph theory and fault diagnosis model. Sections 3 and 4 prove the g-extra diagnosability of the balanced hypercube under the PMC model and MM*

Table 1 Comparative analysis of the ressults of this work and previous studies

Diagnostic model	t_g $(1 \leq g \leq 3)$	t_g $(4 \leq g \leq 5)$	t_g (this paper)
PMC	$4n$-$4+g$ [21]	$6n$-$8+g$ [22]	$\mathbf{2\left[(n-2)\left\lceil\frac{g-1}{2}\right\rceil + n\right] + g}$ $(2 \leq \boldsymbol{g} \leq 2\boldsymbol{n}-1)$ $2^{2\boldsymbol{n}-1} - 1$ $(\boldsymbol{g} = 2\boldsymbol{n})$
MM*	–	–	$\mathbf{2\left[(n-2)\left\lceil\frac{g-1}{2}\right\rceil + n\right] + g}$ $(2 \leq \boldsymbol{g} \leq 2\boldsymbol{n}-1)$ $2^{2\boldsymbol{n}-1} - 1$ $(\boldsymbol{g} = 2\boldsymbol{n})$

model. In Sect. 5, simulation experiments are conducted to verify the correctness of the conclusion. Section 6 concludes the paper.

2 Preliminaries

2.1 Notations

A multiprocessor system is always abstracted as a simple graph $G = (V, E)$, where V denotes the processor set, E denotes communication link set, and $|V|$ denotes the processor number. For any node $v \in V(G)$, we define $N_G(v)$ as the node set adjacent to v, and u is a neighbor node of v where $u \in N_G(v)$. The degree of a node v refers to the number of its neighbor nodes in graph G. Let U be a node set of graph G, and G-U denotes a subgraph by deleting all nodes and all related edges in U from graph G. Given a non-empty node subset $V' \subset V$, the induced subgraph of V' in G is denoted with $G\left[V'\right]$. Let the symmetric difference of two node sets $F_1 \subseteq V(G)$ and $F_2 \subseteq V(G)$ be $F_1 \Delta F_2 = (F_1 - F_2) \cup (F_2 - F_1)$. Let $S \subseteq V(G)$, $N_G(S)$ indicates the set $(\cup_{v \in S} N_G(v)) - S$, $C_G(S)$ indicates the set $N_G(S) \cup S$, and $\overline{S}$ denotes the node set $V(G) - S$. The degree of a node v is denoted with $d_G(v)$, and a graph G is k-regular if for any node v, $d_G(v) = k$. The connectivity (denoted with K_G) of a graph G is the minimum number of removed nodes that may result in the graph G to be disconnected or just one node left [24].

Definition 1 [7] For a graph $G = (V, E)$, a faulty set $F \subseteq V$ is called a g-extra faulty set if every component of $G - F$ has at least $(g + 1)$ nodes.

Definition 2 [23] A g-extra faulty set F is a g-extra cut of G if $G - F$ is disconnected. G is g-extra connected if G has a g-extra cut. The minimum cardinality of g-extra cuts is called the g-extra connectivity for the g-extra connected graph G, denoted by $\tilde{k}_g(G)$.

Definition 3 [8] The g-extra diagnosability of G is the maximum value of t such that G is g-extra t-diagnosable, denoted by $t_g(G)$.

Theorem 1 [8] *A system $G = (V, E)$ is g-extra t-diagnosable under the PMC model if and only if there is an edge $uv \in E$ with $u \in V \backslash \left(F_1 \cup F_2\right)$ and $v \in F_1 \Delta F_2$ for each distinct pair of g-extra subsets F_1 and F_2 of $V\left(BH_n\right)$ with $\left|F_1\right| \le t$ and $\left|F_2\right| \le t$.*

Theorem 2 [25] *For any two distinct faulty subsets F_1 and F_2 in a system $G = (V, E)$, the sets F_1 and F_2 are distinguishable under the PMC model if and only if there exist a node $u \in V \backslash \left(F_1 \cup F_2\right)$ and a node $v \in F_1 \Delta F_2$ such that $uv \in E$ (see Fig. 1).*

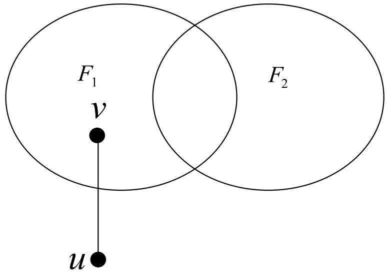

Fig. 1 Two distinguishable sets under the PMC model

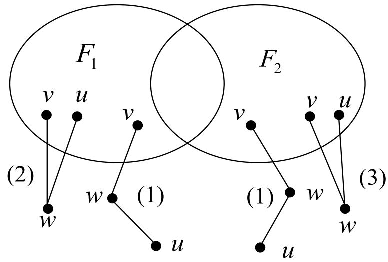

Fig. 2 Two distinguishable sets under the MM* model

Theorem 3 [2] *In a system $G = (V, E)$, G is g-extra t-diagnosable under the MM* model if and only if each distinct pair of g-extra faulty subsets F_1 and F_2 of V with $|F_1| \le t$ and $|F_2| \le t$ satisfies any of the following conditions (see Fig. 2):*

(1) There exist two nodes $u, w \in V \setminus (F_1 \cup F_2)$, and there exists a node $v \in F_1 \Delta F_2$ such that $uw, vw \in E$ (see situation (1) in Fig. 2).
(2) There exist two nodes $u, v \in F_1 \setminus F_2$, and there exists a node $w \in V \setminus (F_1 \cup F_2)$ such that $uw, vw \in E$ (see situation (2) in Fig. 2).
(3) There exist two nodes $u, v \in F_2 \setminus F_1$, and there exists a node $w \in V \setminus (F_1 \cup F_2)$ such that $uw, vw \in E$ (see situation (3) in Fig. 2).

3 The balanced hypercube

For the convenience of proof, we provide the definition of the balanced hypercube as follows.

Definition 4 [26] An n-dimensional balanced hypercube BH_n consists of 2^{2n} nodes $(a_0, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$ where a_0 and $a_i \in \{0, 1, 2, 3\}$ $(1 \le i \le n-1)$. Every node $(a_0, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$ connects the following $2n$ nodes:

(1) $((a_0 \pm 1) \bmod 4, a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, a_{n-1})$;
(2) $((a_0 \pm 1) \bmod 4, a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0}) (\bmod 4), a_{i+1}, \ldots, a_{n-1})$, where i is an integer with $1 \le i \le n-1$.

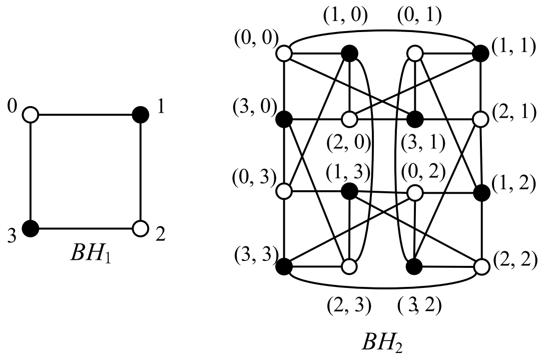

Fig. 3 The structure of BH_1 and BH_2

Definition 5 [26] BH_n is constructed hierarchically as follows (see Fig. 3):

(1) BH_1 is constructed from four nodes connected as a ring. These four nodes are labeled 0, 1, 2, 3, respectively (see subgraph 1 in Fig. 3);

(2) BH_{k+1} is constructed from four BH_ks. These four BH_ks are labeled $BH_k^{(0)}$,$BH_k^{(1)}$,$BH_k^{(2)}$, and $BH_k^{(3)}$, where each node in $BH_k^{(i)}(0 \leq i \leq 3)$ has i attached as the new k-th outer index. Every node $v = (a_0, a_1, \dots a_{k-1}, i)$ in $BH_k^{(i)}(0 \leq i \leq 3)$ has two extra connections (see subgraph 2 in Fig. 3):

(2.1) $BH_k^{(i+1)}$: $(a_0 \pm 1, a_{\cdot 1}, \dots, a_{k-2}, i+1)$ if a_0 is even;

(2.2) $BH_k^{(i-1)}$: $(a_0 \pm 1, a_1, \dots, a_{k-2}, i-1)$ if a_0 is odd.

3.1 The PMC and MM* model

Preparat et al. proposed the PMC faulty diagnosis model in 1967[1], which suppose that each processor has two states, i.e., faulty and faulty-free, and two adjacent nodes can test each other. A test for any two adjacent nodes u and v, performed by u on v, is denoted with an ordered pair (u, v). Suppose the node u is fault-free, the test result is treated as reliable, and the outcome of a test (u, v) is either 1 or 0 [12], where 1 represents that v is a fault node, and vise versa. Conversely, the result is unreliable if the test node u is a fault node [1]. The PMC model can correctly diagnose the state of other processors through the faulty-free processors.

As another classic fault diagnosis model, the MM model assumes that all diagnosis can be implemented by a central processing unit to perform comparison operations and locate faulty processors. The MM model was further generalized to the MM* model by Sengupta and Dahbura in 1992. In the MM* model, a fault-free processor w can perform comparative diagnostic test on v and u if and only if v and u are both the neighbor of w. If w diagnoses that at least one of v and u is in a fault state, the output result is 1, denoted by $\sigma(v, u)_w = 1$; on the contrary, if w diagnoses that u and v are both in a fault-free state, the output result is 0, denoted by $\sigma(v, u)_w = 0$. If the tester w is faulty, the output result is random.

4 The g-extra diagnosability of the balanced hypercube under the PMC model

In this part, we will prove $t_g(G)$ of the balanced hypercube under the PMC model by determining the upper and lower bounds for $2 \le g \le 2n-1$ ($n \ge 2$, n denotes the dimension of the balanced hypercube).

Theorem 4 [17] $\tilde{k}_g\left(BH_n\right) = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right], 2 \le g \le 2n-1.$

For any node $v \in G$, $\{v\} \cup N_G(v)$ forms a connected subgraph G_{sub_c}. it can devide into the following two cases:

(1) (1) If there is no faulty node in G, $|G_{sub\_c}|=2n+1$;
(2) If G_{sub_c} contains at least one faulty node making itself an independent subgraph, by ignoring the faulty nodes, there is $|G_{sub\_c}| \le 2n$. According to the definition of g-extra connectivity, $|G_{sub\_c}| \ge g+1$. So $g+1 \le |G_{sub\_c}| \le 2n$, i.e., $g \le 2n$-1.

Lemma 1 *For arbitrary number* g *where* $2 \le g \le 2n-1 (n \ge 2)$, $t_g\left(BH_n\right) \le 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$ *under the PMC and MM* model.*

Proof Let F be the node set of a connected component $BH_n(F)$ (where $|F| = g+1$), the neighbor node set of F be $N_{BH_n}(F)$, and $N_{BH_n}(F)$ be the minimum g-extra cut of BH_n. Suppose.

$$F_1 = N_{BH_n}(F) \text{ and } F_2 = F \cup F_1$$

By *Theorem 4* and *Definition 2* $|F_1| = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] |F_2| = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + (g+1)$ so the inequality $\left|F_i\right| \le 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + (g+1)$ holds $(i = 1, 2)$. Since $F_1 = N_{BH_n}(F)$ is a g-extra cut of BH_n, each connected component of $BH_n - F_1$ has at least $g+1$ nodes. According to the definition of F_2, every connected component in $BH_n - F_2$ must be contained in $BH_n - F_1$, and also has at least $g+1$ nodes. Thus, by *Definition 1*, F_1 and F_2 are two g-extra faulty sets. From $F_2 = F \cup F_1$, we have $F_1 \Delta F_2 = F$. Note that $N_{BH_n}(F)$ is the neighbor node set of F, and $N_{BH_n}(F) \subseteq F_1 \cup F_2$, there is no connected edge between $F_1 \Delta F_2$ and $V\left(BH_n\right) \setminus \left(F_1 \cup F_2\right)$. From Theorems 1 and 3, it can be drawn that $t_g\left(BH_n\right) \le 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$ under the PMC and MM* model.

Lemma 2 *For arbitrary number* g *where* $2 \le g \le 2n-1 (n \ge 2)$, $t_g\left(BH_n\right) \ge 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$ *under the PMC model.*

Proof Let $t = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$. By *Theorem 1*, in order to prove BH_n is g-extra t-diagnosable under the PMC model, it is necessary to prove that $V\left(BH_n\right) \setminus \left(F_1 \cup F_2\right)$ and $F_1 \Delta F_2$ have at least one connected edge, such that F_1 and F_2 are two distinct g-extra faulty subsets, where $\left|F_1\right| \le t$ and $\left|F_2\right| \le t$.

We will conduct the proof by contradiction. Suppose F_1 and F_2 are two distinct g-extra faulty node sets, where $\left|F_1\right| \leq t = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$ and $\left|F_2\right| \leq t = 2$ $\left|F_2\right| \leq t = 2$, and there is no connected edge between $V\left(BH_n\right)\backslash\left(F_1 \cup F_2\right)$ and $F_1\Delta F_2$. Without loss of generality, we can obtain $F_2\backslash F_1 \neq \emptyset$.

$$V\left(BH_n\right) \neq F_1 \cup F_2$$

Claim 1

Note that $\left|V\left(BH_n\right)\right| = 2^{2n}$, *and for* $2 \leq g \leq 2n-1$, *we have* $\left|F_1 \cup F_2\right| \leq \left|F_1\right| + \left|F_2\right| \leq 2t = 2\left[2(n-2)\lceil\frac{g-1}{2}\rceil + 2n + g\right]$. *Thus,*

$$\begin{aligned}\left|V\left(BH_n\right)\right| - \left|F_1 \cup F_2\right| \geq & 2^{2n} - 2\left[2(n-2)\lceil\frac{g-1}{2}\rceil + 2n + g\right] \\ & 2^{2n} - 2\left[2\left(n^2 - 2n + 2\right) + (2n-1)\right] \\ & = 2^{2n} - 4n^2 + 4n - 6\end{aligned} \tag{1}$$

Let $f(x) = 2^{2x} - 4x^2 + 4x - 6$. $f(x)$ *is obviously a strictly monotonically increasing function when* $x \geq 2$. *We can obtain that* $f(x) \geq f(2) = 2^4 - 16 + 8 - 6 = 2 > 0$. *Thus,* $\left|V\left(BH_n\right)\right| - \left|F_1 \cup F_2\right| > 0$, *and* $V\left(BH_n\right) \neq F_1 \cup F_2$.

Claim 2 $F_1 \cap F_2$ *is a g-extra cut of* BH_n.

*Case 1*$F_1\backslash F_2 \neq \emptyset$.

Since there is no connected edge between $V\left(BH_n\right)\backslash\left(F_1 \cup F_2\right)$ *and* $F_1\Delta F_2$ *(see Fig.* 4), $V(BH_n)\backslash(F_1 \cap F_2)$ *consists of two parts*: $F_1\Delta F_2$ *and* $V\left(BH_n\right)\backslash\left(F_1 \cup F_2\right)$.

Since F_1 *and* F_2 *are two g-extra faulty sets, see Fig.* 5(1) *and* 5(2), *by Definition 1, each connected component in* $V\left(BH_n\right)\backslash F_1$ *has at least* $g+1$ *nodes. Note that there is no connected edge between* $V\left(BH_n\right)\backslash\left(F_1 \cup F_2\right)$ *and* $F_1\Delta F_2$, *then each connected component in* $BH_n\left[F_2\backslash F_1\right]$ *has at least* $g+1$ *nodes, see Fig.* 5 (1). *In the same way,*$BH_n\left[F_1\backslash F_2\right]$ *has at least* $g+1$ *nodes, see Fig.* 5(2). *Therefore, according to Definition 2,* $F_1 \cap F_2$ *is a g-extra cut of* BH_n.

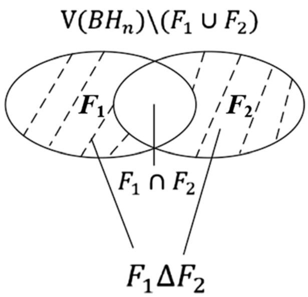

Fig. 4 $V(BH_n)\backslash(F_1 \cap F_2)$ consists of two separate parts

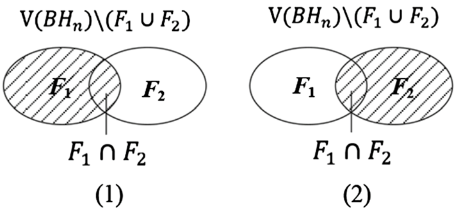

Fig. 5 $F_1 \cap F_2$ is a g-extra faulty set when $F_1 \backslash F_2 \neq \emptyset$

Case 2 $F_1 \backslash F_2 = \emptyset$.

Since $F_1 \backslash F_2 = \emptyset$, *we have* $F_1 \subseteq F_2$. *Then* $F_1 \cap F_2 = F_1$ *is a g-extra faulty set and* $F_1 \Delta F_2 = F_2 \backslash F_1$, *see Fig. 6. Because there is no connected edge between* $\mathrm{V}(BH_n)\backslash(F_1 \cup F_2)$ *and* $F_1 \Delta F_2$, *all the neighbor nodes of* $F_1 \Delta F_2$ *also exist in* $F_1 \cap F_2$. *According to Definition 2*, $F_1 \cap F_2$ *is also a g-extra cut of* BH_n.

By Theorem 4 and the above Claim 2, $|F_1 \cap F_2| \geq 2\left(\lceil \frac{g-1}{2} \rceil n + n - 2\lceil \frac{g-1}{2} \rceil\right)$. *According to what has been discussed in Fig. 5 and Fig. 6, there is* $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \geq (g+1) + 2\left(\lceil \frac{g-1}{2} \rceil n + n - 2\lceil \frac{g-1}{2} \rceil\right) = t + 1$, *a contradiction with* $|F_2| \leq t$. *Therefore*, $t_g(BH_n) \geq 2\left(\lceil \frac{g-1}{2} \rceil n + n - 2\lceil \frac{g-1}{2} \rceil\right) + g$.

Combining Lemma 1 and 2, the following theorem is obtained.

Theorem 5 *For arbitrary number g where* $2 \leq g \leq 2n-1 (n \geq 2)$, $t_g(BH_n) = 2\left[(n-2)\lceil \frac{g-1}{2} \rceil + n\right] + g$ *under the PMC model.*

5 The *g*-extra diagnosability of the balanced hypercube under the MM* model

In this part, we will prove the g-extra diagnosability $t_g(BH_n)$ of the balanced hypercube under the MM* model by determining the upper and lower bounds for $2 \leq g \leq 2n-1$ ($n \geq 3$, n denotes the dimension of the balanced hypercube), i.e., $t_g(BH_n) = 2\left[(n-2)\lceil \frac{g-1}{2} \rceil + n\right] + g$.

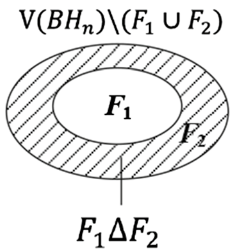

Fig. 6 $F_1 \cap F_2$ is a g-extra faulty set when $F_1 \backslash F_2 = \emptyset$

Lemma 3 *For arbitrary number g where $2 \le g \le 2n-1 (n \ge 3)$, $t_g\left(BH_n\right) \ge 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$ under the MM* model.*

Proof Let $t = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$. We will prove this lemma by contradiction. First, suppose that F_1 and F_2 are two distinguishable g-extra faulty node sets, where $\left|F_1\right| \le t = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$ and $\left|F_2\right| \le t = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$. Obviously, F_1 and F_2 don't meet any of the cases in Theorem 3. According to Claim 1 in Lemma 2, $V\left(BH_n\right) \ne F_1 \cup F_2$.

Claim 1 $F_1 \backslash F_2 \ne \emptyset$, *then* $\left|F_1 \backslash F_2\right| \ge g+1$.

Since $V\left(BH_n\right) \ne F_1 \cup F_2$, we have $\left|V\left(BH_n - F_2\right)\right| \ne \emptyset$. Because F_2 is a g-extra faulty set, each connected component in $BH_n - F_2$ consists of at least $(g+1)$ nodes for $2 \le g \le 2n-1$, then there is no isolated node in $BH_n - F_2$. Moreover, F_1 and F_2 meet none of the cases in Theorem 3, so there exists no connected edge between $V\left(BH_n - F_2 - F_1\right)$ and $V\left(F_1 \backslash F_2\right)$. Note that $BH_n - F_2$ consists of two parts, namely $BH_n - \left(F_1 \cup F_2\right)$ and $V\left(F_1 \backslash F_2\right)$, and F_2 is a g-extra faulty set. According to Definition 1, it's easy to know that each connected component in $V\left(F_1 \backslash F_2\right)$ has at least $(g+1)$ nodes.

Claim 2 *There exists no isolated node in $BH_n - (F_1 \cup F_2)$.*

By contradiction, suppose that there exists at least one orphan node w in $BH_n - \left(F_1 \cup F_2\right)$, it can be divided into the following two sub-cases.

Case 1 $F_1 \backslash F_2 = \emptyset$.

Since $F_1 \backslash F_2 = \emptyset$, we have $F_1 \subseteq F_2$. Because F_1 is a g-extra faulty set and $g \ge 1$, w has at least one neighbor node u such that $u \in F_2 \backslash F_1$. Note that F_1 and F_2 meet none of the conditions in Theorem 3, w has at most one neighbor node $u \in F_2 \backslash F_1$. Thus, there is exactly one neighbor node $u \in F_2 \backslash F_1$ of w. Since F_2 is a g-extra faulty set, each connected component in $BH_n - \left(F_1 \cup F_2\right)$ has at least $(g+1)$ nodes. Therefore,$BH_n - \left(F_1 \cup F_2\right)$ has no isolated node.

Case 2 $F_1 \backslash F_2 \ne \emptyset$.

Let W be the isolated node set in $V\left(BH_n\right) \backslash \left(F_1 \cup F_2\right)$ and let $S = BH_n - \left(F_1 \cup F_2 \cup W\right)$, and $\overline{W} = BH_n - W$. For any node $w \in W$, all neighbor nodes of w are in $F_1 \cup F_2$, that is $N_G(w) \subseteq F_1 \cup F_2$. Since BH_n is n-regular and $\left|V\left(BH_n\right)\right| = 2^{2n}$, we have $\left|\overline{W}\right| * 2n = \sum_{v \in \overline{W}} d_{BH_n}(v) \ge E\left(W, \overline{W}\right) = |W| * 2n$, $|W| \le 2^{2n-1}$.

From $F_1 \backslash F_2 \ne \emptyset$, it's easy to know that $\left|F_1 \backslash F_2\right| \ge 1$, therefore, $\left|F_1 \cap F_2\right| = \left|F_1\right| - \left|F_1 \backslash F_2\right| \le 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g - 1$.

Assume that there exist two nodes $u, v \in F_1 \backslash F_2$ such that $uw, vw \in E\left(BH_n\right)$. By Theorem 3(2), F_1 and F_2 are two distinguishable fault sets, a contradiction to the above assumptions. If there exists no node $u \in F_1 \backslash F_2$ such that $uw \in E(\left(BH_n\right)$, $N_G(w) \subseteq F_2$ and w is an isolated node in $BH_n - F_2$. By Definition 1, we can get

$g = 0$, contradicting with $g \geq 1$. Therefore, there must exist one neighbor node of w in $F_1 \backslash F_2$. The same conclusion that w must have one neighbor node in $F_2 \backslash F_1$ holds (see Fig. 7).

For any node $w \in W$, there are two adjacent vertices $v \in F_1 \backslash F_2$ and $u \in F_2 \backslash F_1$. So, $\left|N_{BH_n}(w) \cap \left(F_2 \cap F_1\right)\right| = \left|N_{BH_n}(w)\right| - \left|N_{BH_n}(w) \cap \left(F_2 \backslash F_1\right)\right| - \left|N_{BH_n}(w) \cap \left(F_1 \backslash F_2\right)\right| = \left|N_{BH_n}(w)\right| - 2 = 2n - 2$, which implies that $\left|F_1 \cap F_2\right| \geq \left|N_{BH_n}(w) \cap \left(F_2 \cap F_1\right)\right| \geq 2n - 2$.

We assume that $V(S) = \emptyset$. Since $\left|F_1 \cap F_2\right| \geq 2n - 2, |W| \leq 2^{2n-1}.,\left|F_i\right| \leq t(i = 1, 2)$ and $2 \leq g \leq 2n - 1$. Then,

$$
\begin{aligned}
2^{2n} = \left|V\left(BH_n\right)\right| &= \left|F_1 \cup F_2\right| + |W| = \left|F_1\right| + \left|F_2\right| - \left|F_1 \cap F_2\right| + |W| \\
&\leq 2\left[2\left[(n-2)\lceil \frac{g-1}{2} \rceil + n\right] + g\right] - (2n-2) + 2^{2n-1} \\
&\leq 2[2(n-1)(n-2) + 4n - 1] - 2(n-1) + 2^{2n-1} \\
&= 2\left(2n^2 - 2n + 3\right) - (2n-2) + 2^{2n-1} \\
&= 4n^2 - 6n + 10 + 2^{2n-1}
\end{aligned} \tag{2}
$$

Let $f(x) = 2^{2x-1} - 4x^2 + 6x - 10$, obviously $f(x)$ is strictly monotonically increasing when $x \geq 3$, so $f(x) \geq f(3) = 4 > 0$, i.e., $2^{2x} \geq 4x^2 - 6x + 10 + 2^{2x-1}$, a contradiction with $2^{2n} \leq 4n^2 - 6n + 10 + 2^{2n-1}$. Therefore, $V(S) \neq \emptyset$.

Because F_1 and F_2 don't satisfy the conditions of (1) and (3) in Theorem 3, and it contains no isolated node in S, we can see that there is no connected edge between $F_1 \Delta F_2$ and S. Since W is the isolated node set in $BH_n - \left(F_1 \cup F_2\right)$, it is impossible to establish one edge between S and W, and hence. $BH_n - \left(F_1 \cap F_2\right)$ is disconnected. Considering that F_1 and F_2 are two g-extra faulty sets, each connected component of $BH_n - F_1$ and $BH_n - F_2$ contains at least $(g+1)$ nodes, and accordingly, each connected component in $BH_n - \left(F_1 \cap F_2\right)$ consists of at least $(g+1)$ nodes as well. So, F_1, F_2 and $F_1 \cap F_2$ are all g-extra cuts.

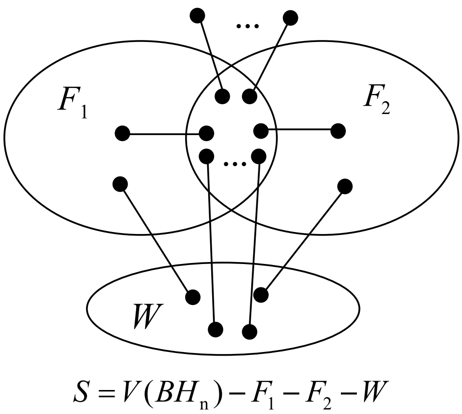

Fig. 7 The relationship between W, S, F_1 and F_2

By Theorem 4, $\left|F_1 \cap F_2\right| \geq 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right]$ and $\left|F_1\right| \leq 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$, we have $\left|F_1 \backslash F_2\right| = \left|F_1\right| - \left|F_1 \cap F_2\right| \leq 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g - 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] \leq g$. We can get $\left|F_2 \backslash F_1\right| \leq g$ in the same way, a contradiction with Claim 1. The poof of Claim 2 is finished.

According to Claim 2, $BH_n - \left(F_1 \cup F_2\right)$ consists of no isolated node. It's easy to know that, for $\forall\ u \in BH_n - \left(F_1 \cup F_2\right)$, u has at least one neighbor in $BH_n - \left(F_1 \cup F_2\right)$. Since F_1 and F_2 meet none of the conditions in Theorem 3, u has no neighbor in $F_1 \Delta F_2$. Due to the arbitrariness of u, there is no connected edge between $BH_n - \left(F_1 \cup F_2\right)$. and $F_1 \Delta F_2$, so $BH_n - \left(F_1 \cap F_2\right)$ is disconnected. Because F_1 and F_2 are two faulty sets, each connected component in $BH_n - F_1$ and $BH_n - F_2$ has at least $(g+1)$ nodes, accordingly, each connected component in $BH_n - \left(F_1 \cap F_2\right)$ has at least $(g+1)$ nodes as well. According to Definition 2, $F_1 \cap F_2$ is a g-extra cut of BH_n. By Theorem 4, s$\left|F_1 \cap F_2\right| \geq 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right]$.

On the other hand, since $F_1 \cap F_2$ is a g-extra cut of BH_n, $\left|F_1 \backslash F_2\right| \geq g+1$, so $\left|F_1\right| = \left|F_1 \cap F_2\right| + \left|F_1 \backslash F_2\right| \geq t+1$, contradicting with the assumption that $\left|F_1\right| \leq t = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$. The conclusion of this lemma holds.

According to Lemma 1 and Lemma 3, the following Theorem 6 obviously holds.

Theorem 6 *For arbitrary number g where $2 \leq g \leq 2n-1 (n \geq 3)$, then $t_g\left(BH_n\right) = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$ under the MM* model.*

The discussions above just explored the g-extra diagnosability of BH_n under the PMC and MM* model, respectively, for $2 \leq g \leq 2n-1$. Then we will consider another special situation when $g = 2n$.

Theorem 7 [27] *Let S_1 and S_2 be a distinct pair of g-extra faulty subsets of $V(G)$ in a system $G=(V, E)$. If $S_1 \cup S_2 = V$, (S_1, S_2) is indistinguishable under the PMC model and MM* model.*

Theorem 8 *For $g = 2n$, the g-extra diagnosability of BH_n under the PMC and MM* model is $t_g\left(BH_n\right) = 2^{2n-1} - 1$.*

Proof. Let F_1 and F_2 be two distinct g-extra faulty sets, $F_1 = V(BH_{n-1}^{(0)} + BH_{n-1}^{(1)})$ and $F_2 = V(BH_{n-1}^{(3)} +$, then $F_1 = F_2 = 2^{2n-1}$. By Theorem 7, F_1 and F_2 are indistinguishable. By Theorem 1 and 3, $t_g\left(BH_n\right) \leq 2^{2n-1} - 1$.

(1) According to Theorem 1, in order to prove that BH_n is g-extra t-diagnosable under the PMC model, we need to first prove that there is at least one connected edge between $V\left(BH_n\right) \backslash \left(F_1 \cup F_2\right)$ and $F_1 \Delta F_2$ for each distinct pair of g-extra faulty subsets F_1 and F_2, where $\left|F_i\right| \leq 2^{2n-1} - 1 (i = 1, 2)$.

Since $\left|F_1\right| \le 2^{2n-1}-1$ and $\left|F_2\right| \le 2^{2n-1}-1$, we can get $V(BH_n)\backslash(F_1 \cup F_2) \neq \emptyset$, i.e., $\exists u \in V(BH_n)\backslash(F_1 \cup F_2)$. For two distinct g-extra faulty sets F_1 and F_2, without loss of generality, we might as well suppose $F_1\backslash F_2 \neq \emptyset$, then $\exists v \in F_1\backslash F_2$. Note that BH_n is complete and connected, uv is an edge connecting $V(BH_n)\backslash(F_1 \cup F_2)$ and $F_1 \Delta F_2$. By Theorem 1, $t_g(BH_n) = 2^{2n-1}-1$.

(2) According to Theorem 3, in order to prove that BH_n is g-extra t-diagnosable under the MM* model, it is necessary to prove that there are three nodes u, v, and w that satisfy one of three conditions in Theorem 3 for each distinct pair of g-extra faulty subsets F_1 and F_2, where $\left|F_i\right| \le 2^{2n-1}-1(i=1,2)$.

Since $\left|F_1\right| \le 2^{2n-1}-1$ and $\left|F_2\right| \le 2^{2n-1}-1$, we can get $V(BH_n)\backslash(F_1 \cup F_2) \neq \emptyset$ and $V(BH_n)\backslash(F_1 \cup F_2) \ge 2$, then there are two nodes $u, w \in V(BH_n)\backslash(F_1 \cup F_2)$. Similar to (1), we suppose $F_1\backslash F_2 \neq \emptyset$, then $\exists v \in F_1\backslash F_2$. Since BH_n is complete and connected, we can have that $uw \in E(BH_n)$, and uv is an edge connecting $V(BH_n)\backslash(F_1 \cup F_2)$ and $F_1 \Delta F_2$. By Theorem 3, $t_g(BH_n) = 2^{2n-1}-1$.

Combining Theorems 5, 6, and 8, we summarize the following Theorem 9 s on the g-extra diagnosability of the balanced hypercube.

Theorem 9 *The g-extra diagnosability of the balanced hypercube under the PMC model for $n \ge 2$ and the MM* model for $n \ge 3$ is*

$$t_g = \begin{cases} 4n-3, & g = 1(\textit{under the PMC model}) \\ 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g, & 2 \le g \le 2n-1 \\ 2^{2n-1}-1, & g = 2n \end{cases}.$$

Where the conlusion of the situation $g=1$ is quoted from reference [21].

6 Simulation and analysis

In this part, we conduct simulation experiments to verify the correctness of g-extra diagnosability of BH_n. The main flow of the simulation experiment is: (1) connect to the graph database neo4j, and establish the BH_n network; (2) randomly generate faulty sets $F_k(k=1,2,\ldots)$; (3) judge whether F_k is a g-extra faulty set; (4) judge whether F_m and F_n $(m, n=1,2,\ldots)$ are diagnosable according to Theorem 2 and Theorem 3. The simulation experiment flow chart is shown in Fig. 8.

6.1 Simulation experiment environment

This experimental environment configurations and development tools are listed in Table 2.

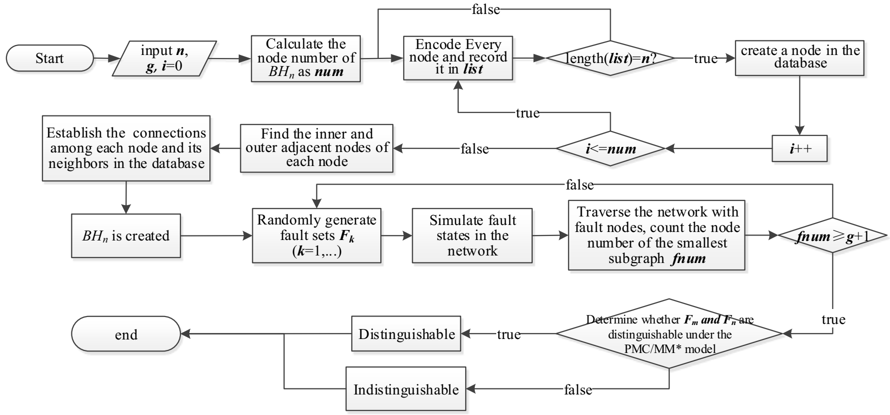

Fig. 8 Simulation experimental flowchart

Table 2 Experimental Environment Configurations and Development Tools

Platform attribute	Details
RAM	180G
CPU	Intel(R) Xeon(R) Gold 5120 CPU @ 2.20 GHz 32-core processor
GPU	NVIDIA Corporation Device 1eb8 (rev a1)
Operating System	Linux 3.10.04
Development tools	Python-3.8, neo4j-community-4.3.3, JDK11
Runtime environment	python3, JDK 11 or above
Development languages	Python, Cyper

6.2 Distinguishable Decision Algorithms and Computational complexity analysis

6.2.1 Distinguishable Decision algsorithms under the PMC and MM* model

(1) Distinguishable decision algorithm under the PMC model.

The algorithm starts by traversing every node in $F_{s1} \Delta F_{s2}$, then finds all neighbor nodes of each node, and judges whether all the neighbor nodes are in $V \backslash \left(F_{s1} \cup F_{s2}\right)$. If there exists one neighbor node that is not in $V \backslash (F_{s1} \cup F_{s2})$, F_{s1} and F_{s2} are distinguishable. According to Theorem 2, the system is g-extra t-diagnosable under the PMC model.

Algorithm 1. Distinguishable Decision Algorithm under the PMC Model
Input: g-extra faulty sets F_{s1} and F_{s2}
Output: whether F_1 and F_2 are distinguishable under the PMC model

```
Fs1 Δ Fs2 = {};
Fs1 ∪ Fs2 = {};
for node in Fs1
    if node not in Fs2
        Fs1 Δ Fs2 ∪ {node};
        Fs1 ∪ Fs2 ∪ {node};
    end if
end for
for node in Fs2
    if node not in Fs1
        Fs1 Δ Fs2 ∪ {node};
    end if
    Fs1 ∪ Fs2 ∪ {node};
end for
for node in Fs1 Δ Fs2
    NeighborNodes = FindNeighbor(node);
    for oneNeighbor in NeighborNodes
        if oneNeighbor not in Fs1 ∪ Fs2
            return true;
        end if
    end for
end for
return false;
```

(2) Distinguishable decision algorithm under the MM* model.

The algorithm starts by traversing every node in $V\backslash(F_{s1} \cup F_{s2})$, then finds all neighbor nodes of each node, and judges whether there exists one neighbor node that satisfies any of the conditions in Theorem 3. If there is such a neighbor node, F_{s1} and F_{s2} are distinguishable. According to Theorem 3, the system is g-extra t-diagnosable under the MM* model.

Algorithm 2: Distinguishable Decision Algorithm under the MM* model
Input: g-extra faulty set F_1 and F_2
Output: whether F_1 and F_2 are distinguishable under the MM* model

```
Count1 = 0;
Count2 = 0;
Count3 = 0;
Count4 = 0;
for node in V\(Fs1 ∪ Fs2)
    NeighborNodes = FindNeighbor(node);
    for oneNeighbor in NeighborNodes
        if oneNeighbor in Fs1\Fs2
            Count1++;
        if oneNeighbor in Fs2\Fs1
            Count2++;
        if oneNeighbor in Fs1ΔFs2
            Count3++;
        if oneNeighbor in V\(Fs1 ∪ Fs2)
            Count4++;
    end for
    if Count1>=2 or Count2>=2 or (Count3>=1 and Count4>=1)
        return true;
        break;
    end if
end for
return false;
```

6.2.2 Computational complexity analysis

We now discuss the computational complexity of distinguishable decision algorithms. Under the PMC model, the computation is mainly consumed in judging whether the neighbor nodes of all $F_{s1}\Delta F_{s2}$ nodes are in $V\backslash(F_{s1}\cup F_{s2})$. So, it's easy to get the computational complexity under the PMC model, i.e., $O(|F_{s1}\Delta F_{s2}|*2n)$. Under the MM* model, the computation is mainly consumed in judging whether all neighbor nodes satisfy one of the conditions in the double layer for loop. Similarly, the computational complexity can be obtained, i.e., $O(|V\backslash(F_{s1}\cup F_{s2})|*2n)$.

6.2.3 Application example

In this part, we take BH_3 as the example to illustrate the applicability of g-extra diagnosability and proposed algrithms.

Example1 Given the network BH_3, and let $g=5$. In this example, we randomly generate a distinct pair of g-extra faulty sets F_{m1} and F_{m2}, and explain how to judge whether F_{m1} and F_{m2} are distinguishable, so as to determine whether the system is 5-extra t-diagnosable under the MM* model.

Step 1 Calculate $t_5\left(BH_3\right)$ according to the formula $t_g\left(BH_n\right) = 2\left[(n-2)\lceil\frac{g-1}{2}\rceil + n\right] + g$ in Theorem 9, i.e., $t_5\left(BH_3\right)=11$.

Step 2 Let $\left|F_{m1}\right| \le t_5\left(BH_3\right)$, $|F_{m2}| \le t_5\left(BH_3\right)$, and randomly generate two g-extra faulty sets as follows:

$$F_{m1} = \{ (1,1,2), (2,1,1), (2,0,2), (3,2,1), (2,2,1), (1,3,1), (0,3,1), (3,3,1), (2,3,1), (3,3,2), (2,3,2)\}$$

$$F_{m2} = \{ (2,2,1), (3,3,1), (0,3,0), (1,3,1), (3,2,2), (2,3,2), (1,0,1), (0,3,1), (3,0,1), (2,0,0)(1,1,0)\}$$

Step 3 Randomly select one node (2,0,2) from F_{m1}, and calculate all its neighbor nodes $N_G((2,0,2))$. We can find a node $(3,0,2) \in N_G((2,0,2))$ that satisfies $(3,0,2) \notin F_{m1}\Delta F_{m2}$, and there exists one neighbor node (0,0,1)$\in V\backslash(F_{\mathrm{m1}} \cup F_{\mathrm{m2}})$ (see Fig. 9). By Theorem 3, it is known that F_{m1} and F_{m2} are distinguishable. Similarly, there also exists one node (3,2,2) in F_{m2} that makes F_{m1} and F_{m2} distinguishable.

Since F_{m1} and F_{m2} are distinguishable, according to Theorem 3, we can draw the conclusion that the system is 5-extra 11-diagnosable.

6.3 Simulation experiments and results analysis

To validate the correctness of the g-extra diagnosabilities of BH_n proposed above, we carry out a series of simulation experiments by setting different values to g under different network dimensions n, and then make judgements on whether two randomly chosen faulty sets are distinguishable, shown in Table 3 and Table 4, with the last column indicating judgement results. Owing to the limitations of network scale and computational complexity (BH_{14} contains more than 100 million nodes), we conduct the experiments from different node orders of magnitude with $n=14$ as the maximum network dimension.

Since the diagnosabilities of BH_n under the PMC model have been verified in previous research, we only consider the cases $g \ge 6$ in this paper.

Based on Theorem 9, we define the binary function:

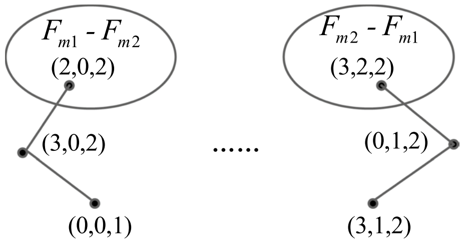

Fig. 9 Some special nodes making F_{m1} and F_{m2} distinguishable

Table 3 Simulation experiment results under the PMC model

Dimension n	g ($g \le 2n$-1)	Number of nodes	Number of edges	t_g	Judge result
4	6	256	1048	26	True
(*hundred level*)	7	256	1048	27	True
	8	256	1048	31	True
5	6	1024	5120	34	True
(*thousand level*)	…	…	…	…	…
	9	1024	5120	43	True
6	6	4096	24,579	42	True
(*thousand level*)	…	…	…	…	…
	11	4096	24,579	63	True
7	6	16,384	114,688	50	True
(*tens of thousand level*)	…	…	…	…	…
	13	16,384	114,688	87	True
8	6	65,536	524,288	58	True
(*tens of thousand*	…	…	…	…	…
level)	15	65,536	524,288	115	True
9	6	262,144	2,359,296	66	True
(*hundreds of thousand*	…	…	…	…	…
level)	17	262,144	2,359,296	183	True
10	6	1,048,576	10,485,760	74	True
(*million level*)	…	…	…	…	…
	19	1,048,576	10,485,760	223	True
12	6	16,777,216	201,326,596	90	True
(*tens of million level*)	…	…	…	…	…
	23	16,777,216	201,326,596	267	True
14	6	268,435,456	3,758,096,284	106	True
(*hundreds of million level*)	…	…	…	…	…
	27	268,435,456	3,758,096,284	315	True

$$z = 2\left[(y-2)\lceil\frac{x-1}{2}\rceil + y\right] + x \quad (2 \le x \le 2y-1)$$

with x and y, respectively, corresponding to g and n in Theorem 9, and z representing the value of g-extra diagnosability. In function z, the partial derivatives of the independent variables x and y are always positive. It can be observed from the three-dimensional function image in Fig. 10 that, the function value z increases with the increase of x or y. Therefore, it is concluded that the g-extra diagnosability of BH_n increases with the growth of the value g, or with the expanding of network scale n, which is a preferable property for the fault diagnosis of large scale BH_n networks.

Table 4 Simulation experiment results under the MM* model

Dimension n	g ($g \leq 2n$-1)	Number of nodes	Number of edges	t_g	Judge result
4	2	256	1024	14	True
(*hundred level*)	…	…	…	…	…
	7	256	1024	27	True
5	2	1024	5120	18	True
(*thousand level*)	…	…	…	…	…
	9	1024		43	True
6	2	4096	24,576	22	True
(*thousand level*)	…	…	…	…	…
	11	4096	24,576	63	True
7	2	16,384	114,688	26	True
(*tens of thousand level*)	…	…	…	…	…
	13	16,384	114,688	87	True
8	2	65,536	524,288	30	True
(*tens of thousand level*)	…	…	…	…	…
	15	65,536	524,288	115	True
9	2	262,144	2,359,296	34	True
(*hundreds of thousand*	…	…	…	…	…
level)	17	262,144	2,359,296	147	True
10	2	1,048,576	10,485,790	38	True
(*million level*)	…	…	…	…	…
	19	1,048,576	10,485,790	183	True
11	2	4,194,304	4,194,304	42	True
(*million level*)	…	…	…	…	…
	19	4,194,304	4,194,304	223	True
12	2	16,777,216	201,326,592	46	True
(*tens of million level*)	…	…	…	…	…
	23	16,777,216	201,326,592	267	True
14	2	268,435,456	3,758,096,284	54	True
(*hundreds of million level*)	…	…	…	…	…
	27	268,435,456	3,758,096,284	315	True

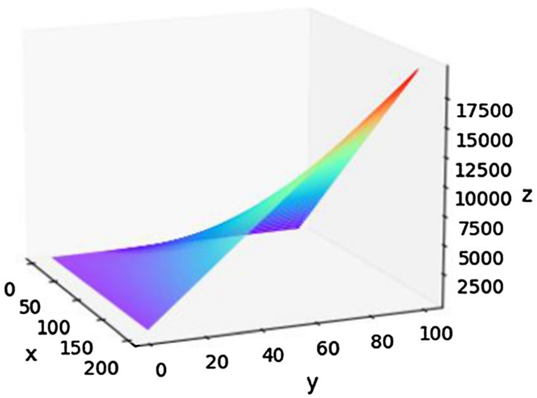

Fig. 10 The g-extra diagnosability of BH_n increases with the growth of the value g or the network scale n

7 Conclusion

In this paper, we, respectively, proved that $t_g(BH_n) = 2\left[(n-2)\left\lceil \frac{g-1}{2} \right\rceil + n\right] + g$ for $2 \le g \le 2n-1$ under the PMC and MM* model, and $t_g(BH_n) = 2^{2n-1} - 1$ for $g = 2n$. At the same time, simulation experiment has been done to verify the correctness of the conclusions. Compared to existing research literatures, our result extends the existing g-extra diagnosabilty of the balanced hypercube under the PMC model to a more general condition, i.e., $2 \le g \le 2n$. What's more, we also obtained the g-extra diagnosabilty of the balanced hypercube under the MM* model, which is seldomly involved in current research literatures. In the end, simulation experiments are conducted to validate the effectiveness our proposed theories. Following the research in this paper, we will further focus on the g-extra diagnosabilities of BC networks and other regular networks in our future research work.

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