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Can a minimal degree 6 cubature rule for the disk have all points inside?

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Abstract

We use positivity and extension properties of moment matrices to prove that a 10-node (minimal) cubature rule of degree 6 for planar measure on the closed unit disk $\bar{\mathbb{D}}$ cannot have all nodes in $\bar{\mathbb{D}}$. We construct examples showing that such rules may have as many as 9 points in $\bar{\mathbb{D}}$, and we provide similar examples for the triangle.

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1. Introduction

Let μ be a positive Borel measure on $\mathbb{R}^2$ having convergent moments up to at least degree m, i.e., $\beta_{ij} \equiv \int x^i y^j \, d\mu$ is absolutely convergent for all $i, j \geqslant 0$ satisfying $i + j \leqslant m$. A *cubature rule* for μ of *degree m* and *size N* consists of *nodes* $(x_1, y_1), \ldots, (x_N, y_N)$ in $\mathbb{R}^2$ and positive weights $\rho_1, \ldots, \rho_N$

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such that $\int p(x, y)\,d\mu(x, y) = \sum_{k=1}^{N} \rho_k\, p(x_k, y_k)$ for each polynomial $p \in \mathbb{R}[x, y]$ with deg $p \leqslant m$; equivalently, $\nu \equiv \sum_{k=1}^{N} \rho_k\, \delta_{(x_k, y_k)}$ satisfies $\beta_{ij} = \int x^i y^j\, d\nu$ for all $i, j \geqslant 0$ with $i + j \leqslant m$. (Here, $\delta_{(x_k, y_k)}$ denotes the unit-mass measure supported at (x_k, y_k).) The cubature rule ν is *minimal* if size$(\nu) \leqslant$ size(ω) whenever ω is a cubature rule for μ of degree *m*, and ν is an *inside rule* if supp $\nu \subset$ supp μ.

Two recurrent themes in cubature literature are the estimation of the fewest nodes possible in a cubature rule of prescribed degree, and the construction of corresponding minimal rules (cf. [2,16,24,28]). Even for Lebesgue measure on basic planar sets such as the disk or square, minimal rules or minimal inside rules are known only for small values of *m* (cf., [3,5]). In the case when $\mu = \mu_{\bar{\mathbb{D}}}$, planar measure on the closed unit disk $\bar{\mathbb{D}}$, minimal rules are known only for $m \leqslant 7$ and $m = 9$. For $m \leqslant 5$ and $m = 7$, there are minimal rules that are also inside rules (cf. [3,5]). For $m = 6$, minimal rules have 10 points, and all such rules documented in the literature have at least 2 points outside; rules with 8 points inside are documented in [14,22,26]. This suggests the question discussed in [3, p. 26], as to whether there exists an inside rule with as few as 10 nodes. Our main result resolves this question as follows.

Theorem 1.1. *There is no degree* 6 *minimal inside rule for* $\mu_{\bar{\mathbb{D}}}$ *with as few as* 10 *nodes.*

In Section 4 we construct 10 node rules of degree 6 for $\mu_{\bar{\mathbb{D}}}$ with 9 points inside $\bar{\mathbb{D}}$; in view of Theorem 1.1, these rules are optimal among minimal rules with respect to the number of nodes inside the disk. It is known that minimal (respectively, minimal inside) rules for $\mu_{\bar{\mathbb{D}}}$ of degree 7 have 12 nodes (cf. [2–4]), but it remains an open question whether there exists a degree 6 inside rule for $\mu_{\bar{\mathbb{D}}}$ with as few as 11 nodes. At least in principle, our techniques can be adapted to minimality questions for other measures, regions, or degrees. In each situation, however, one needs to solve systems of nonlinear equations and inequalities, or demonstrate that these systems admit no solution. The proof of Theorem 1.1 is based on the latter. As for the former, in addition to our results in Section 4 for the disk, in Section 5 we present some new cubature rules for the triangle T_2. The size of a minimal inside rule of degree 6 is unknown, but in Section 5 we construct several new 10-node minimal degree 6 rules with 9 points inside.

Before discussing our approach to the proof of Theorem 1.1, we briefly digress to place our results in a wider context. Minimal cubature rules for strictly positive integrals on the real line enjoy the following features associated with classical Gaussian quadrature: (i) for each $m > 0$, there exists a minimal rule with precisely dim $\mathscr{P}_{[m/2]}$ nodes (where $\mathscr{P}_k \equiv \mathbb{R}_k[x] = \{p \in \mathbb{R}[x]\colon \text{ deg } p \leqslant k\}$), (ii) for $m = 2n$, there are infinitely many such minimal rules, precisely one of which interpolates all moments of degree $2n + 1$, (iii) the nodes are the zeros of a quasiorthogonal polynomial, (iv) the weights are positive, (v) for each $m > 0$, there is a minimal rule that is also an inside rule. The extent to which these properties characterize minimal cubature rules in higher dimensions has been the subject of much study (cf., [2]). For certain measures on $\mathbb{R}^n$, minimal cubature rules do exhibit the above Gaussian properties [1]. On the other hand, classical integrals, such as those induced by Lebesgue measure on the *n*-ball, *n*-cube, or *n*-simplex in $\mathbb{R}^n$, generally fail to have minimal cubature rules of degree *m* with as few as dim $\mathbb{R}^n_{[m/2]}[x]$ nodes (cf., [6,16,17,19,20,25]). Theorem 1.1 apparently provides the first evidence that property (v) may fail for a classical integral in higher dimensions.

The proof of Theorem 1.1 is given in Section 3, and is based on the development of multivariable cubature in [12], utilizing positivity and extension properties of moment matrices. For cubature problems in the plane, this method permits us to analyze cubature rules using either moments for monomials $x^i y^j$ (as above), or using moments for complex monomials $\bar{z}^i z^j$. In the proof of Theorem 1.1 and in the examples of Sections 4 and 5 we will use complex monomials and corresponding complex moment matrices. We next recall the main features of this approach; for a discussion of real and complex moment matrices in arbitrarily many variables, see [10]. In particular, all of the properties and theorems concerning moment matrices that we present in the remainder of this section extend to multivariable moment matrices [10].

Given a complex sequence $\gamma \equiv \gamma^{(2n)} = \{\gamma_{ij} : i, j \geqslant 0, \ i+j \leqslant 2n\}$ and a closed set $K \subset \mathbb{C}$, the *truncated complex K-moment problem* for γ entails determining whether there exists a positive Borel measure ω on $\mathbb{C}$ such that

$$\gamma_{ij} = \int_{\mathbb{C}} \bar{z}^i z^j \, \mathrm{d}\omega(z), \quad (i, j \geqslant 0, \ i + j \leqslant 2n) \tag{1.1}$$

and

$$\operatorname{supp} \omega \subset K; \tag{1.2}$$

a measure satisfying (1.1) is a *representing measure* for γ; ω is a *K-representing measure* if it satisfies (1.1) and (1.2). If μ is a positive Borel measure on $\mathbb{R}^2$ having convergent moments with respect to $x^i y^j$ up to at least degree $2n$, we may also view μ as a measure on $\mathbb{C}$ having convergent moments up to at least degree $2n$ with respect to $\bar{z}^i z^j$. A cubature rule for μ thus corresponds to a *finitely atomic* representing measure for $\gamma^{(2n)}[\mu]$ (with $\gamma_{ij} = \int \bar{z}^i z^j \, \mathrm{d}\mu$) (cf., [9, Proposition 1.12]). Such a cubature rule v is an inside rule if v is a K-representing measure for $K = \operatorname{supp} \mu$.

Given $\gamma \equiv \gamma^{(2n)}$ as above, we associate to γ the *complex moment matrix* $M(n) \equiv M(n)(\gamma)$ as follows. Let $\mathbb{C}_n[z, \bar{z}]$ denote the space of complex polynomials in z and $\bar{z}$ of degree at most n. For $p \in \mathbb{C}_n[z, \bar{z}]$, with $p(z, \bar{z}) \equiv \sum_{r,s \geqslant 0, \ r+s \leqslant n} a_{rs} \bar{z}^r z^s$, let $\hat{p} \equiv (a_{rs})$; $\hat{p}$ is the coefficient vector of p relative to the basis for $\mathbb{C}_n[z, \bar{z}]$ consisting of the monomials $\{\bar{z}^i z^j : i, j \geqslant 0, \ i + j \leqslant n\}$, ordered degree-lexicographically as $1, z, \bar{z}, z^2, z\bar{z}, \bar{z}^2, \ldots$. We recall the *Riesz functional* $\Lambda \equiv \Lambda_\gamma : \mathbb{C}_{2n}[z, \bar{z}] \to \mathbb{C}$ defined by $\Lambda(\sum_{r,s \geqslant 0, \ r+s \leqslant 2n} b_{rs} \bar{z}^r z^s) = \sum b_{rs} \gamma_{rs}$. The matrix $M(n)$, of size $\dim \mathbb{C}_n[z, \bar{z}] (= (n+1)(n+2)/2)$, is uniquely determined by

$$\langle M(n)(\gamma) \hat{f}, \hat{g} \rangle = \Lambda_\gamma(f \bar{g}), \quad f, g \in \mathbb{C}_n[z, \bar{z}]. \tag{1.3}$$

If γ has a representing measure v, then $\Lambda_\gamma(f\bar{g}) = \int f \bar{g} \, \mathrm{d}v$; in particular $\langle M(n) \hat{f}, \hat{f} \rangle = \Lambda_\gamma(|f|^2) = \int |f|^2 \, \mathrm{d}v \geqslant 0$. So $M(n)$ is positive semidefinite in this case. For the case of complex moment matrices in the plane, we will give a concrete description of $M(n)$ in Section 2.

If μ is a representing measure for γ, then $\operatorname{card} \operatorname{supp} \mu \geqslant \operatorname{rank} M(n)$ [7, Corollary 3.7]. The main result of [7] shows that γ has a rank $M(n)$-atomic (i.e., *minimal*) representing measure if and only if $M(n) \geqslant 0$ and $M(n)$ can be extended to a moment matrix $M(n+1)$ satisfying $\operatorname{rank} M(n+1) = \operatorname{rank} M(n)$ (i.e., $M(n+1)$ is a *flat extension* of $M(n)$) [7, Theorem 5.13]. In this case, $M(n+1)$ is itself positive, and $M(n+1)$ admits unique successive (positive) flat moment matrix extensions $M(n+2), M(n+3), \ldots, M(\infty)$. Further, $M(\infty)$ admits a unique representing measure, $\tilde{v}$, which is rank $M(n)$-atomic [7, Corollary 5.12]. Conversely, if $\tilde{v}$ is a rank $M(n)$-atomic representing measure for $M(n)$, then $M(n+1)[\tilde{v}]$ is a flat extension of $M(n)$, with unique successive flat moment matrix extensions $M(n+2)[\tilde{v}], M(n+3)[\tilde{v}], \ldots$.

Suppose $M(n)$ is positive and admits a flat extension $M(n+1)$. Let $K \equiv K_Q$ be the semi-algebraic subset of $\mathbb{C}$ defined by a finite family of polynomials $Q = \{q_i\}_{i=1}^m \subset \mathbb{C}[z, \bar{z}]$, i.e., $K_Q = \{z \in \mathbb{C} : q_i(z, \bar{z}) \geqslant 0, \ 1 \leqslant i \leqslant m\}$. Let $\deg q_i = 2k_i$ or $2k_i - 1$. Relative to the flat extension $M(n + k_i)$ (op. cit.), there is a *localizing matrix* $M_{q_i}(n + k_i)$ (see Section 2 and [8, Section 3]). Our main tool in proving Theorem 1.1 is the following characterization of rank $M(n)$-atomic K_Q-representing measures.

Theorem 1.2 (*Curto and Fialkow [8, Theorem 1.1]*). *$\gamma \equiv \gamma^{(2n)}$ has a* rank $M(n)$*-atomic K_Q-representing measure if and only if $M(n) \geqslant 0$ and $M(n)$ admits a flat extension $M(n+1)$ for which $M_{q_i}(n+k_i) \geqslant 0$, $1 \leqslant i \leqslant m$.*

For planar Lebesgue measure on a semialgebraic set K_Q with nonempty interior, cubature rules of degree $2n$ with as few as rank $M(n)$ atoms exist only for small values of n. In general, a minimal cubature rule would correspond to a positive extension $M(n+p)$ (for some $p \geqslant 0$) such that $M(n+p)$ admits a flat extension $M(n+p+1)$ for which $M_{q_i}(n+p+k_i) \geqslant 0$, $(1 \leqslant i \leqslant m)$ [8] (cf., [12]). For generalizations of Theorem 1.2 to higher dimensions, see [10]. To study cubature rules of degree 6 for $\mu_{\bar{\mathbb{D}}}$, we will utilize $\gamma \equiv \gamma^{(6)}[\mu_{\bar{\mathbb{D}}}]$ and $M(3) \equiv M(3)[\mu_{\bar{\mathbb{D}}}]$. Now rank $M(3) = 10$ (cf., Section 2). As described above, flat extensions $M(4)$ correspond to 10-atomic representing measures $\tilde{v}$, which in turn correspond to cubature rules for $\mu_{\bar{\mathbb{D}}}$ of degree 6. Note that $\bar{\mathbb{D}}$ is the semi-algebraic set corresponding to $q(z,\bar{z}) = 1 - z\bar{z}$. To prove Theorem 1.1,we will show that if $M(4)$ is a flat extension of $M(3)$, then $\operatorname{supp} \tilde{v} \not\subset \bar{\mathbb{D}}$, and for this we will use Theorem 1.2, and show that $M_q(4)$ is not positive semidefinite.

In Sections 4 and 5 we compute certain cubature rules associated with flat extensions, using the method that we next describe. We denote the successive columns of $M(n)$ by

$$1, Z, \bar{Z}, Z^2, Z\bar{Z}, \bar{Z}^2, \ldots, Z^n, Z^{n-1}\bar{Z}, \ldots, Z\bar{Z}^{n-1}, \bar{Z}^n.$$

Corresponding to a polynomial $p(z,\bar{z}) \equiv \sum a_{ij}\bar{z}^i z^j \in \mathbb{C}_n[z,\bar{z}]$ we have the element $p(Z,\bar{Z})$ in $\operatorname{Col} M(n)$, the column space of $M(n)$, defined by $p(Z,\bar{Z}) = \sum a_{ij}\bar{Z}^i Z^j$. Thus each column dependence relation in $M(n)$ may be expressed as $p(Z,\bar{Z}) = 0$ for some $p \in \mathbb{C}_n[z,\bar{z}]$. We define the *variety* of $M(n)$ by

$$\mathscr{V}(M(n)) = \bigcap_{\substack{p \in \mathbb{C}_n[z,\bar{z}] \\ p(Z,\bar{Z})=0}} \mathscr{Z}(p),$$

where $\mathscr{Z}(p) = \{z \in \mathbb{C} : p(z,\bar{z}) = 0\}$. Suppose μ is a representing measure for $\gamma^{(2n)}$. It follows from [7, Proposition 3.1] that

$$\text{for } p \in \mathbb{C}_n[z,\bar{z}], \quad \operatorname{supp}\mu \subset \mathscr{Z}(p) \Leftrightarrow p(Z,\bar{Z}) = 0, \tag{1.4}$$

whence

$$\operatorname{supp}\mu \subset \mathscr{V}(M(n)). \tag{1.5}$$

Moreover, from [7, Corollary 3.7],

$$\operatorname{rank} M(n) \leqslant \operatorname{card} \operatorname{supp}\mu \leqslant \operatorname{card} \mathscr{V}(M(n)). \tag{1.6}$$

Now suppose $M(n)$ is positive and invertible and has a flat extension $M(n+1)$. We will compute the unique representing measure $\tilde{v}$ for $M(\infty)$ described above. Since rank $M(n+1) = $ rank $M(n)$, for $0 \leqslant i \leqslant n+1$, there exists $p_i \in \mathbb{C}_n[z,\bar{z}]$ such that $\bar{Z}^i Z^{n+1-i} = p_i(Z,\bar{Z})$ in $\operatorname{Col} M(n+1)$. Let $q_i(z,\bar{z}) = \bar{z}^i z^{n+1-i} - p_i(z,\bar{z}) \in \mathbb{C}_{n+1}[z,\bar{z}]$. Let $r = \operatorname{rank} M(n)$ $(= \dim \mathbb{C}_n[z,\bar{z}])$ and suppose $\mathscr{V} \equiv \bigcap_{i=0}^{n+1} \mathscr{Z}(q_i)$ $(\supseteq \mathscr{V}(M(n+1)))$ has exactly r points, say $\mathscr{V} = \{z_1, \ldots, z_r\}$. Since card supp $\tilde{v} = r$, (1.6) implies $r = \operatorname{rank} M(n) = \operatorname{rank} M(n+1) \leqslant \operatorname{card} \operatorname{supp} \tilde{v} \leqslant \operatorname{card} \mathscr{V}(M(n+1)) \leqslant \operatorname{card} \mathscr{V} = r$, whence (1.5) implies that $\operatorname{supp} \tilde{v} = \mathscr{V}$.Thus, $\tilde{v} = \sum_{i=1}^r \rho_i \delta_{z_i}$ for certain positive densities ρ_i. To compute these, let V be the $r \times r$

matrix, with rows labeled $1, Z, \bar{Z}, Z^2, Z\bar{Z}, \bar{Z}^2, \ldots, \bar{Z}^i Z^j, \ldots, Z^n, \ldots, \bar{Z}^n$, such that row $\bar{Z}^i Z^j$ equals $(\bar{z}_1^i z_1^j, \ldots, \bar{z}_r^i z_r^j)$. Since $M(n)$ is invertible, (1.4) (applied to $\mu = \tilde{v}$) implies that V is invertible. Since $\tilde{v}$ is a representing measure for $\gamma^{(2n)}$, we have $V(\rho_1, \ldots, \rho_r)^{\mathrm{t}} = (\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \ldots, \gamma_{0n}, \ldots, \gamma_{n0})^{\mathrm{t}}$, which uniquely determines $\rho_1, \ldots, \rho_r$.(Above we computed $\tilde{v}$ under the assumption that card $\mathscr{V} = r$; results of [10] show that this is always the case. Indeed, [10, Theorem 1.2] shows that if $M(n+1)$ is a flat extension of $M(n)$, then supp $\tilde{v} = \mathscr{V}(M(n+1))$, whence card $\mathscr{V}(M(n+1)) =$ card supp $\tilde{v} =$ rank $M(n) = r$. Moreover, since $M(n)$ is invertible, $\mathscr{V} = \mathscr{V}(M(n+1))$, whence card $\mathscr{V} = r$. The results of this note are independent of [10], which also treats the case when $M(n)$ is singular.)

Since supp $\mu_{\bar{\mathbb{D}}}$ has nonempty interior, (1.4) implies that $M(3)[\mu_{\bar{\mathbb{D}}}]$ is invertible with rank 10. In Section 4, independent of [10, Theorem 1.2], we exhibit flat extensions $M(4)$ of $M(3)[\mu_{\bar{\mathbb{D}}}]$ such that $\mathscr{V}$ has exactly 10 points, 9 of which lie in $\bar{\mathbb{D}}$. The measure $\tilde{v}$ (as computed above) thus acts as a 10-node cubature rule for $\mu_{\bar{\mathbb{D}}}$ of degree 6 having 9 points inside $\bar{\mathbb{D}}$.

2. Moment matrices and localization

In this section we give a concrete description of planar moment matrices and localizing matrices. Consider a complex sequence γ of degree $2n$, i.e., $\gamma \equiv \gamma^{(2n)} = \{\gamma_{ij} : i, j \geqslant 0, \ i + j \leqslant 2n\}$. Recall that $M(n) \equiv M(n)(\gamma)$ is defined as the unique matrix (of size $(n+1)(n+2)/2$) such that

$$\langle M(n)(\gamma)\hat{f}, \hat{g}\rangle = \Lambda_\gamma(f\bar{g}), \quad f, g \in \mathbb{C}_n[z, \bar{z}], \tag{2.1}$$

where $\Lambda_\gamma(\cdot)$ is the Riesz functional associated to γ (cf., Section 1). Now size $M(n) = \dim \mathbb{C}_n[z, \bar{z}]$, and we denote the rows and columns of $M(n)$ as

$$1, Z, \bar{Z}, Z^2, Z\bar{Z}, \bar{Z}^2, \ldots, Z^n, Z^{n-1}\bar{Z}, \ldots, Z\bar{Z}^{n-1}, \bar{Z}^n$$

ordered by the degree-lexicographic ordering of the monomials in $\mathbb{C}_n[z, \bar{z}]$. In particular, it follows from (2.1) that the entry of $M(n)$ in row $\bar{Z}^k Z^l$, column $\bar{Z}^i Z^j$ is equal to

$$\langle M(n)\widehat{\bar{z}^i z^j}, \widehat{\bar{z}^k z^l}\rangle = \gamma_{i+l, j+k} \ (i, j, k, l \geqslant 0, \ \ i + j, k + l \leqslant n). \tag{2.2}$$

For example, with $n = 1$, the *quadratic moment problem* for $\gamma \equiv \gamma^{(2)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$ corresponds to

$$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix}$$

with rows and columns $1, Z, \bar{Z}$.

We next give a block matrix decomposition of $M(n)$ that is convenient for discussing moment matrix extensions. For $0 \leqslant i, \ j \leqslant n$, let $M[i, j]$ denote the $(i+1) \times (j+1)$ matrix whose entries are the moments of order $i + j$, as follows:

$$M[i, j] = \begin{pmatrix} \gamma_{ij} & \gamma_{i+1, j-1} & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1, j+1} & \gamma_{ij} & \cdots & \gamma_{i+j-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{0,i+j} & \gamma_{1,i+j-1} & \cdots & \gamma_{ji} \end{pmatrix}.$$

A straightforward calculation using (2.2) shows that

$$M(n) = (M[i, j])_{0 \leqslant i, j \leqslant n}. \tag{2.3}$$

For example, with $n = 1$ we have

$$M(1) = \begin{pmatrix} M[0,0] & M[0,1] \\ M[1,0] & M[1,1] \end{pmatrix}.$$

If $\gamma \equiv \gamma^{(2n)}$ corresponds to the moments of a positive Borel measure μ supported in $\mathbb{C}$, then $M(n)$ is positive semidefinite, and hence self-adjoint. So $\bar{\gamma}_{ij} = \gamma_{ji}$, and in this case we sometimes write $M(n)$ as $M(n)[\mu]$. In the sequel we consider the case when μ denotes normalized planar measure on the closed unit disk $\mathbb{D}$, i.e., $\mu = \mu_0 \equiv (1/\pi)\mu_{\bar{\mathbb{D}}}$, so that $\gamma_{ii} = 1/(i+1)$, $0 \leqslant i \leqslant n$, and $\gamma_{ij} = 0$ for $i, j \geqslant 0$, $i + j \leqslant 2n$, $i \neq j$. The proof of Theorem 1.1 depends on calculations with $M(3)[\mu_0]$, which assumes the form

$$M(3)[\mu_0] = \begin{pmatrix}
1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}
\end{pmatrix}. \tag{2.4}$$

In the sequel we also consider a positive semidefinite moment matrix extension $M(4)$ of $M(3) = M(3)[\mu_0]$, which assumes the form

$$M(4) = \begin{pmatrix} M(3) & B \\ B^* & C \end{pmatrix}, \tag{2.5}$$

where

$$B = \begin{pmatrix} M[0,4][\mu_0] \\ M[1,4][\mu_0] \\ M[2,4][\mu_0] \\ M[3,4] \end{pmatrix}$$

and $C = M[4, 4]$. More concretely,

$$B = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ x & \overline{x} & \overline{y} & \overline{z} & \overline{w} \\ y & x & \overline{x} & \overline{y} & \overline{z} \\ z & y & x & \overline{x} & \overline{y} \\ w & z & y & x & \overline{x} \end{pmatrix}, \tag{2.6}$$

where x, y, z, and w are complex numbers corresponding to moments of degree 7, i.e., $x = \gamma_{34}$, $y = \gamma_{25}$, $z = \gamma_{16}$, $w = \gamma_{07}$ and we set $\gamma_{ij} = \bar{\gamma}_{ji}$. Further, C is a self-adjoint (indeed, positive semidefinite) Toeplitz matrix of the form

$$C = \begin{pmatrix} \gamma_{44} & \gamma_{53} & \gamma_{62} & \gamma_{71} & \gamma_{80} \\ \gamma_{35} & \gamma_{44} & \gamma_{53} & \gamma_{62} & \gamma_{71} \\ \gamma_{26} & \gamma_{35} & \gamma_{44} & \gamma_{53} & \gamma_{62} \\ \gamma_{17} & \gamma_{26} & \gamma_{35} & \gamma_{44} & \gamma_{53} \\ \gamma_{08} & \gamma_{17} & \gamma_{26} & \gamma_{35} & \gamma_{44} \end{pmatrix}. \tag{2.7}$$

We next describe *localizing matrices* which play a central role in the proof of Theorem 1.1. For $p \in \mathbb{C}_{2n}[z, \bar{z}]$, let $d \equiv \deg\, p = 2k$ or $2k - 1$ $(0 \leqslant k \leqslant n)$, and set $\sigma \equiv \sigma(k, n) = \dim \mathbb{C}_{n-k}[z, \bar{z}]$ $(=(n - k + 1)(n - k + 2)/2)$. The map $\varphi_p : \mathbb{C}^\sigma \times \mathbb{C}^\sigma \to \mathbb{C}$ defined by $\varphi_p(\hat{f}, \hat{g}) = \Lambda_\gamma(pf\bar{g})$ $(f, g \in \mathbb{C}_{n-k}[z, \bar{z}])$ is sesquilinear. Thus, by the Riesz Representation Theorem for sesquilinear forms, there exists a unique matrix $M_p(n) \equiv M_p(n)(\gamma)$ of size σ such that

$$\langle M_p(n)\hat{f}, \hat{g}\rangle = \varphi_p(\hat{f}, \hat{g})(=\Lambda_\gamma(pf\bar{g}))(f, g \in \mathbb{C}_{n-k}[z, \bar{z}]). \tag{2.8}$$

Let $\mathscr{K}_p = \{z \in \mathbb{C} : p(z, \bar{z}) \geqslant 0\}$. Note that if there exists a representing measure μ for γ with $\operatorname{supp} \mu \subset \mathscr{K}_p$, then $\langle M_p\hat{f}, \hat{f}\rangle = \Lambda_\gamma(p|f|^2) = \int p|f|^2\, d\mu \geqslant 0$ $(f \in \mathbb{C}_{n-k}[z, \bar{z}])$, whence $M_p \geqslant 0$.

We next provide a concrete description of $M_p(n)(\gamma)$. Let us write $p(z, \bar{z}) = \sum_{r,s\geqslant 0,\ 0\leqslant r+s\leqslant 2k} a_{rs}\bar{z}^r z^s$. Note that for each $r, s \geqslant 0$ with $r + s \leqslant 2k$, there exist $i, j \geqslant 0$ such that $0 \leqslant i + j \leqslant k$, $0 \leqslant (r + s) - (i + j) \leqslant k$. Thus $\bar{z}^r z^s = \bar{z}^i z^j \cdot \bar{z}^{r-i} z^{s-j}$ with $\deg\, \bar{z}^i z^j$, $\deg\, \bar{z}^{r-i} z^{s-j} \leqslant k$. (The preceding decomposition is not unique.) Let $M_{rs} \equiv [\bar{z}^{s-j} z^{r-i}; \sigma]M(n)[\bar{z}^i z^j; \sigma]$ denote the compression of $M(n)$ to the first σ columns that are indexed by multiples of $\bar{z}^i z^j$ and to the first σ rows indexed by multiples of $\bar{z}^{s-j} z^{r-i}$. (M_{rs} is independent of the choice of i and j in the decomposition of $\bar{z}^r z^s$ [8, Lemma 3.4].)

Theorem 2.1 (*Curto and Fialkow [8, Theorem 3.5]*). $M_p(n) = \sum_{0\leqslant r+s\leqslant d} a_{rs} M_{rs}$.

In the sequel, we are concerned with the case $n = 4$ and $p(z, \bar{z}) = 1 - z\bar{z}$. In this case, we have $k = 1$, $\sigma = 10$. To compute $M_p(4)$, we first note from the uniqueness of M_p satisfying (2.8) that $M_p(4) = [M_1(4)]_{10} - M_{z\bar{z}}(4)$. Now the polynomial 1 corresponds to $r = s = i = j = 0$; thus $M_1(4) = M(4)$, whence $[M_1(4)]_{10} = M(3)$. Consequently,

$$M_p(4) = M(3) - M_{z\bar{z}}(4). \tag{2.9}$$

For the monomial $z\bar{z}$, we have $r = s = 1$. So we may take $i = 0,\ j = 1$, whence $M_{z\bar{z}}(4)$ is the central compression of $M(4)$ to the first 10 rows and columns that are multiples of Z (the rows and columns $Z, Z^2, \bar{Z}Z, Z^3, Z^2\bar{Z}, Z\bar{Z}^2, Z^4, Z^3\bar{Z}, Z^2\bar{Z}^2, Z\bar{Z}^3$). Thus,

$$M_{z\bar{z}}(4) = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{51} \\ \gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{15} & \gamma_{24} & \gamma_{33} & \gamma_{42} \\ \gamma_{31} & \gamma_{32} & \gamma_{41} & \gamma_{33} & \gamma_{42} & \gamma_{51} & \gamma_{34} & \gamma_{43} & \gamma_{52} & \gamma_{61} \\ \gamma_{22} & \gamma_{23} & \gamma_{32} & \gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{25} & \gamma_{34} & \gamma_{43} & \gamma_{52} \\ \gamma_{13} & \gamma_{14} & \gamma_{23} & \gamma_{15} & \gamma_{24} & \gamma_{33} & \gamma_{16} & \gamma_{25} & \gamma_{34} & \gamma_{43} \\ \gamma_{41} & \gamma_{42} & \gamma_{51} & \gamma_{43} & \gamma_{52} & \gamma_{61} & \gamma_{44} & \gamma_{53} & \gamma_{62} & \gamma_{71} \\ \gamma_{32} & \gamma_{33} & \gamma_{42} & \gamma_{34} & \gamma_{43} & \gamma_{52} & \gamma_{35} & \gamma_{44} & \gamma_{53} & \gamma_{62} \\ \gamma_{23} & \gamma_{24} & \gamma_{33} & \gamma_{25} & \gamma_{34} & \gamma_{43} & \gamma_{26} & \gamma_{35} & \gamma_{44} & \gamma_{53} \\ \gamma_{14} & \gamma_{15} & \gamma_{24} & \gamma_{16} & \gamma_{25} & \gamma_{34} & \gamma_{17} & \gamma_{26} & \gamma_{35} & \gamma_{44} \end{pmatrix}. \tag{2.10}$$

Since the structure of $M(4)$ is given by (2.5), using (2.4), (2.6), and (2.7) we can write

$$M_{z\bar{z}}(4) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & x & \bar{x} & \bar{y} & \bar{z} \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{4} & 0 & y & x & \bar{x} & \bar{y} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & z & y & x & \bar{x} \\ 0 & 0 & 0 & \bar{x} & \bar{y} & \bar{z} & \gamma_{44} & \gamma_{53} & \gamma_{62} & \gamma_{71} \\ 0 & \frac{1}{4} & 0 & x & \bar{x} & \bar{y} & \gamma_{35} & \gamma_{44} & \gamma_{53} & \gamma_{62} \\ 0 & 0 & \frac{1}{4} & y & x & \bar{x} & \gamma_{26} & \gamma_{35} & \gamma_{44} & \gamma_{53} \\ 0 & 0 & 0 & z & y & x & \gamma_{17} & \gamma_{26} & \gamma_{35} & \gamma_{44} \end{pmatrix}. \tag{2.11}$$

3. On the 10-point degree 6 cubature rule for the disk

In this section we prove Theorem 1.1. We begin by analyzing the structure of a flat (i.e., rank-preserving) moment matrix extension $M(4)$ of $M(3) \equiv M(3)[\mu_0]$. Such an extension $M(4)$ has the form described by (2.5)–(2.7). To ensure that $M(4)$ is positive semidefinite and that $\operatorname{rank} M(4) = \operatorname{rank} M(3)[\mu_0]$, it is necessary and sufficient that

$$C = C^{\sharp} \equiv B^* M(3)^{-1} B \tag{3.1}$$

(cf., [8, (2.4)]). Now $C^{\sharp}$ is positive semidefinite (hence self-adjoint) and [7, Proposition 2.3] implies that each diagonal of $C^{\sharp}$ is symmetric about its midpoint. Thus, if $C^{\sharp} = (c_{ij})_{1 \leqslant i,j \leqslant 5}$, the main diagonal is of the form $d_1 : \ c_{11}, c_{22}, c_{33}, c_{22}, c_{11}$ and the successive diagonals below d_1 are of the form $d_2 : c_{21}, c_{32}, c_{32}, c_{21}; d_3 : c_{31}, c_{42}, c_{31}; d_4 : c_{41}, c_{41}; d_5 : c_{51}$. Whence $C^\sharp$ is completely determined by

$$\begin{array}{lll} c_{11} & & \\ c_{21} & c_{22} & \\ c_{31} & c_{32} & c_{33}. \\ c_{41} & c_{42} & \\ c_{51} & & \end{array} \tag{3.2}$$

A calculation of $C^\sharp$ shows that (3.2) is equal to

$$\begin{array}{lll} 4\,\bar{x}x + 36\,\bar{y}y + 36\,\bar{z}z + 4\,\bar{w}w, & & \\ 4\,x^2 + 36\,\bar{x}y + 36\,\bar{y}z + 4\,\bar{z}w, & \frac{3}{16} + 40\,\bar{x}x + 36\,\bar{y}y + 4\,\bar{z}z, & \\ 40\,yx + 36\,\bar{x}z + 4\,\bar{y}w, & 40\,\bar{x}y + 36\,x^2 + 4\,\bar{y}z, & \frac{7}{36} + 8\,\bar{y}y + 72\,\bar{x}x, \\ 40\,zx + 36\,y^2 + 4\,\bar{x}w, & 8\,\bar{x}z + 72\,yx, & \\ 8\,wx + 72\,zy. & & \end{array} \tag{3.3}$$

Recall from (3.1) that we require that $C^\sharp$ have the form of a moment matrix block $C \equiv C(4)$ as in (2.7). In particular, we require that $C^\sharp$ be Toeplitz, i.e., constant on each diagonal. Since c_{11}, c_{22}, and c_{33} must have a common value, say α, we have

$$\begin{aligned} &4|x|^2 + 36|y|^2 + 36|z|^2 + 4|w|^2 = \alpha, \\ &\tfrac{3}{16} + 40|x|^2 + 36|y|^2 + 4|z|^2 = \alpha, \\ &\tfrac{7}{36} + 72|x|^2 + 8|y|^2 = \alpha. \end{aligned} \tag{3.4}$$

Further, we may express $c_{21} = c_{32}$ and $c_{31} = c_{42}$ as

$$4x^2 + 36\bar{x}y + 36\bar{y}z + 4\bar{z}w = 40\bar{x}y + 36x^2 + 4\bar{y}z \tag{3.5}$$

and

$$40yx + 36\bar{x}z + 4\bar{y}w = 8\bar{x}z + 72yx. \tag{3.6}$$

Simplifying (3.5) and (3.6), we obtain

$$7\bar{x}z + \bar{y}w = 8xy \tag{3.7}$$

and

$$8\bar{y}z + \bar{z}w = \bar{x}y + 8x^2. \tag{3.8}$$

Also, subtracting the first 2 equations of (3.4) yields

$$32|z|^2 + 4|w|^2 = \tfrac{3}{16} + 36|x|^2, \tag{3.9}$$

while subtracting first and third equations of (3.4) yields

$$28|y|^2 + 36|z|^2 + 4|w|^2 = \tfrac{7}{36} + 68|x|^2. \tag{3.10}$$

Numerical experiments readily show that $M(3)[\mu_0]$ admits infinitely many flat extensions $M(4)$, and for each such extension the moments of degree 7 satisfy (3.7)–(3.10).

From (2.9), $M_p(4) = M(3) - M_{z\bar{z}}(4)$, where $M(3) \equiv M(3)[\mu_0]$ is given by (2.4). To evaluate $M_{z\bar{z}}(4)$, we take the general form given in (2.11) and set $\gamma_{44} = \alpha\,(=c_{11} = c_{22} = c_{33})$ (cf., (3.4)), $\gamma_{35} \equiv \bar{\beta}\,(=c_{21} = c_{32})$ (cf., (3.5)), $\gamma_{26} \equiv \bar{\gamma}\,(=c_{31} = c_{42})$ (cf., (3.6)), and $\gamma_{17} \equiv \bar{\delta}\,(=c_{41})$. We now obtain

$$M_p(4) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 \\ 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & -x & -\bar{x} & -\bar{y} & -\bar{z} \\ \frac{1}{6} & 0 & 0 & 0 & \frac{1}{12} & 0 & -y & -x & -\bar{x} & -\bar{y} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & -z & -y & -x & -\bar{x} \\ 0 & 0 & 0 & -\bar{x} & -\bar{y} & -\bar{z} & \frac{1}{4}-\alpha & -\beta & -\gamma & -\delta \\ 0 & \frac{1}{12} & 0 & -x & -\bar{x} & -\bar{y} & -\bar{\beta} & \frac{1}{4}-\alpha & -\beta & -\gamma \\ 0 & 0 & \frac{1}{12} & -y & -x & -\bar{x} & -\bar{\gamma} & -\bar{\beta} & \frac{1}{4}-\alpha & -\beta \\ 0 & 0 & 0 & -z & -y & -x & -\bar{\delta} & -\bar{\gamma} & -\bar{\beta} & \frac{1}{4}-\alpha \end{pmatrix}.$$

Let us partition $M_p(4)$ as

$$M_p(4) = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{B}^* & \tilde{C} \end{pmatrix},$$

where $\tilde{A}$ is the 6×6 block in the upper left corner. It is easy to verify that $\tilde{A}$ is positive and invertible. Choleski's algorithm [11] thus implies that $M_p(4) \geqslant 0$ if and only if $\varDelta \equiv \tilde{C} - \tilde{B}^*\tilde{A}^{-1}\tilde{B} \geqslant 0$, and a calculation shows that the entry in row 2, column 2 of $\varDelta$ is equal to

$$\varDelta_{22} \equiv \tfrac{5}{24} - \alpha - 48|x|^2 - 12|y|^2.$$

Thus, if $M(4)$ is a flat extension for which $M_p(4) \geqslant 0$, then $\varDelta_{22} \geqslant 0$, and using the third condition in (3.4), we may express the last inequality as

$$120|x|^2 + 20|y|^2 \leqslant \tfrac{1}{72}. \tag{3.11}$$

In view of the previous exposition (including Theorem 1.2 and the remarks following it), to prove Theorem 1.1 it suffices to establish the following result.

Proposition 3.1. *If $M(4)$ is a flat extension of $M(3)[\mu_0]$, then $M_{1-z\bar{z}}(4)$ is not positive semi-definite*; *in particular, the system consisting of Eqs.* (3.7)–(3.10) *and inequality* (3.11) *has no solution.*

We will prove Proposition 3.1 by using a chain of simple estimates. From now on, x, y, z, and w will be assumed to satisfy (3.7)–(3.10) as well as (3.11), and we will derive a contradiction. We start with a collection of identities.

Lemma 3.2.

(a) $|z|^2 = \frac{1}{32}(\frac{3}{16} + 36|x|^2 - 4|w|^2)$.
(b) $|x|^2 = \frac{1}{32}(28|y|^2 + 4|z|^2 - \frac{1}{144})$.
(c) $|z|^2 = \frac{1}{4}(32|x|^2 - 28|y|^2 + \frac{1}{144})$.
(d) $|w|^2 = \frac{1}{4}(224|y|^2 - 220|x|^2 + \frac{19}{144})$.

Proof. Formula in (a) is just Eq. (3.9) solved for $|z|^2$. Part (b) follows by subtracting (3.9) from (3.10). To prove (c), just solve (b) for $|z|^2$. For (d), combine (a) and (c) to get

$$\frac{1}{32}(\frac{3}{16} + 36|x|^2 - 4|w|^2) = \frac{1}{4}(32|x|^2 - 28|y|^2 + \frac{1}{144});$$

simplifying and solving for $|w|^2$ yields the desired result. □

Next we prove several estimates.

Lemma 3.3.

(a) $120|x|^2 + 20|y|^2 \leqslant \frac{1}{72}$.
(b) $125|y|^2 + 15|z|^2 \leqslant \frac{23}{576}$.
(c) $46|x|^2 \leqslant 9|y|^2 + 2|z|^2$.
(d) $|x|^2 \leqslant \frac{1}{7\times 220}$.
(e) $|w|^2 \geqslant \frac{23}{6\times 144}$.

Proof. Part (a) is (3.11). From (a) and Lemma 3.2(b) we get

$$\frac{120}{32}(28|y|^2 + 4|z|^2 - \frac{1}{144}) + 20|y|^2 \leqslant \frac{1}{72};$$

simplifying this we get (b). To prove part (c), we notice that, by Lemma 3.2(c), we have

$$8|z|^2 - 64|x|^2 + 56|y|^2 = \frac{1}{72}.$$

Combining with (a), we get $120|x|^2 + 20|y|^2 \leqslant 8|z|^2 - 64|x|^2 + 56|y|^2$, from which the conclusion easily follows. Part (d) is a direct consequence of (a), since $|y|^2 \geqslant 0$. Finally, we prove (e). Since $|y|^2 \geqslant 0$, the equality established in Lemma 3.2(d) implies that

$$|w|^2 \geqslant \frac{1}{4}(\frac{19}{144} - 220|x|^2).$$

Using (d), we get the desired estimate. □

Now we will prove the crucial link in the chain that will lead to the proof of Proposition 3.1.

Lemma 3.4. $|x| < \frac{4}{5}|y|$.

Proof. Suppose to the contrary that

$$|y| \leqslant \frac{5}{4}|x|. \tag{3.12}$$

Using the triangle inequality, (3.8) implies that $|z|\,|w| \leqslant 8|x|^2 + |x|\,|y| + 8|y|\,|z|$. In view of (3.12), we get

$$|z|\,|w| \leqslant 8|x|^2 + |x|\,\tfrac{5}{4}\,|x| + 8|z|\,\tfrac{5}{4}\,|x| = \tfrac{37}{4}\,|x|^2 + 10|x|\,|z|. \tag{3.13}$$

Next, we combine (3.12) and Lemma 3.3(c). Then $46|x|^2 \leqslant 9|x|^2\, 25/16 + 2|z|^2$, so $|x|^2 \leqslant 32/511\,|z|^2$. If we use this estimate in (3.13), we see that

$$|z|\,|w| \leqslant \frac{37}{4}\frac{32}{511}\,|z|^2 + 10|z||z|\sqrt{\frac{32}{511}} = |z|^2\left(\frac{296}{511} + 10\sqrt{\frac{32}{511}}\right)$$

and, consequently,

$$|w| \leqslant |z| 3.0817\ldots < |z|\sqrt{10}. \tag{3.14}$$

Using (3.14) in (3.9) we get $3/16 + 36|x|^2 > 4|w|^2 + 32\,|w|^2/10 = 36/5|w|^2$. Now estimates obtained in Lemma 3.3(d) and (e) yield

$$\frac{23}{120} = \frac{36}{5}\frac{23}{6\times 144} \leqslant \frac{36}{5}|w|^2 < \frac{3}{16} + 36|x|^2 \leqslant \frac{3}{16} + 36\frac{1}{72\times 120} = \frac{3}{16} + \frac{1}{240} = \frac{23}{120}.$$

This contradiction shows that $|x| < 4/5|y|$. □

The inequality established in Lemma 3.4 will now allow us to improve the estimates from Lemma 3.3(d) and (e).

Lemma 3.5.

(a) $|x| < \frac{1}{104}$;
(b) $|w| \geqslant \sqrt{19}/24$.

Proof. By Lemma 3.4, $|y| > 5/4\,|x|$. So (3.11) implies that

$$\tfrac{1}{72} > 120|x|^2 + 20 \times \tfrac{25}{16}\,|x|^2 = \tfrac{605}{4}\,|x|^2,$$

whence

$$|x|^2 < \frac{4}{72\times 605} = \frac{1}{10890} < \frac{1}{10816} = \frac{1}{104^2},$$

which proves (a). To prove (b), we notice that Lemma 3.4 implies that $|x| < |y|$, so that $224|y|^2 - 220|x|^2 > 0$. Therefore, Lemma 3.2(d) shows that $4|w|^2 > 19/144$, and the result follows. □

Lemma 3.4 established that $|x| < |y|$. Now we compare $|y|$ and $|z|$.

Lemma 3.6. $|y| < 0.7|z|$.

Proof. The triangle inequality applied to (3.7) yields

$$|y|\,|w| \leqslant 7|x|\,|z| + 8|x|\,|y|. \tag{3.15}$$

By Lemma 3.5(b), $|w| \geqslant \sqrt{19}/24 = 0.181620\ldots > 0.1730\ldots = 18/104$. Using this estimate and Lemma 3.5(a), inequality (3.15) implies that

$$\tfrac{18}{104}|y| < (7|z| + 8|y|)\,\tfrac{1}{104}.$$

A simple calculation completes the proof. □

So far we have established estimates on $|x|$ and $|w|$. Now we give an upper bound for $|y|$.

Lemma 3.7. $|y| < 0.0161$.

Proof. By Lemma 3.6, $|y| < 0.7|z|$, whence $|z|^2 > \frac{100}{49}|y|^2 > 2|y|^2$. We will use this inequality in the estimate obtained in Lemma 3.3(b). Notice that $23/576 < 155 \times 0.0161^2$. Therefore, $125|y|^2 + 30|y|^2 < 155 \times 0.0161^2$, and the conclusion follows. □

The following estimate provides the contradiction which completes the proof of Proposition 3.1, and, consequently, also completes the proof of Theorem 1.1.

Lemma 3.8. $|y| \geqslant 0.0161$.

Proof. We start with a simple consequence of (3.8):

$$|z|\,|w| \leqslant 8|y|\,|z| + |x|\,|y| + 8|x|^2. \tag{3.16}$$

Lemmas 3.4 and 3.6 together imply that $|x| < 0.56|z|$ (so, in particular, $|z| > 0$). This estimate and (3.16) show that $|z|\,|w| \leqslant 8|y|\,|z| + |y|(0.56|z|) + 8|x|(0.56|z|)$, and thus $|w| \leqslant 8|y| + 0.56|y| + 8(0.56)|x|$.We conclude that

$$8.56|y| \geqslant |w| - 8(0.56)|x|. \tag{3.17}$$

By Lemma 3.5(b), $|w| \geqslant \sqrt{19}/24 > 2.36/13$. This estimate, Lemma 3.5(a), and (3.17) yield

$$8.56|y| > \frac{2.36}{13} - 8(0.56)\,\frac{1}{104} = \frac{2.36}{13} - \frac{0.56}{13} = \frac{1.8}{13} > \frac{8.56}{62}.$$

Therefore, $|y| > 1/62 = 0.016129\ldots > 0.0161$. □

4. 10-Node degree 6 rules with 9 nodes inside $\bar{\mathbb{D}}$

In this section we compute several 10-node degree 6 cubature rules for the disk, each having 9 nodes inside $\bar{\mathbb{D}}$. In view of Theorem 1.1, these rules are optimal with respect to the number of nodes inside $\bar{\mathbb{D}}$. As discussed in Sections 1 and 2, 10-node, degree 6 cubature rules for $\mu_{\bar{\mathbb{D}}}$ correspond to flat extensions $M(4)$ of $M(3) \equiv M(3)[\mu_{\bar{\mathbb{D}}}]$, i.e.,

$$M(4) = \begin{pmatrix} M(3) & B(4) \\ B(4)^* & C(4) \end{pmatrix}, \tag{4.1}$$

where $W \equiv M(3)^{-1}B(4)$ satisfies

$$C(4) = W^* M(3)^{-1} W. \tag{4.2}$$

$M(4)$ is thus completely determined by a choice of "new moments" of degree 7, which we denote as $\gamma_{34} = x \equiv x_1 + ix_2$, $\gamma_{25} = y \equiv y_1 + iy_2$, $\gamma_{16} = u \equiv u_1 + iu_2$, $\gamma_{07} = w \equiv w_1 + iw_2$ (with $\gamma_{ij} = \bar{\gamma}_{ji}$). A calculation shows that (4.2) is equivalent to the system

$$\text{test}\, i = 0 \quad (i = 1, \ldots, 6), \tag{4.3}$$

where $\text{test}\,1 = -3\pi^2 + 64|w|^2 - 576|x|^2 + 512|u|^2$; $\text{test}\,2 = -\pi^2 - 4608|x|^2 + 4032|y|^2 + 576|u|^2$; $\text{test}\,3 = 8x_1^2 - 8x_2^2 + x_1y_1 + x_2y_2 - w_1u_1 - 8y_1u_1 - w_2u_2 - 8y_2u_2$; $\text{test}\,4 = 16x_1x_2 - x_2y_1 + x_1y_2 - w_2u_1 + 8y_2u_1 + w_1u_2 - 8y_1u_2$; $\text{test}\,5 = w_1y_1 - 8x_1y_1 + w_2y_2 + 8x_2y_2 + 7x_1u_1 + 7x_2u_2$; $\text{test}\,6 = w_2y_1 - 8x_2y_1 - w_1y_2 - 8x_1y_2 - 7x_2u_2 + 7x_1u_2$. (This system is analogous to (3.4)–(3.6), except that we are now using $\mu_{\bar{\mathbb{D}}}$, not μ_0, and we have simplified some tests by clearing denominators.)

For a flat extension $M(4)$ of $M(3)$, we next compute $\mathscr{V} \equiv \bigcap_{i=0}^{4} \mathscr{Z}(q_i)$ (cf. Section 1). Moment matrix structure dictates that $q_0 = \bar{q}_4$ and $q_1 = \bar{q}_3$ [7, Lemma 3.10]. So we have, more simply, $\mathscr{V} = \bigcap_{i=0}^{2} \mathscr{Z}(q_i)$. Denoting the 10 rows of W degree-lexicographically, as $1, Z, \bar{Z}, Z^2, Z\bar{Z}, \bar{Z}^2, Z^3, Z^2\bar{Z}, Z\bar{Z}^2, \bar{Z}^3$, let $W_{\bar{Z}^i Z^j, k}$ denote the entry of W in row $\bar{Z}^i Z^j$, column k. Then (4.1) and (4.2) imply that $q_k(z, \bar{z}) = \bar{z}^k z^{4-k} - \sum_{i,j \geqslant 0,\ i+j \leqslant 3} W_{\bar{Z}^i Z^j, k} \bar{z}^i z^j$, $0 \leqslant k \leqslant 4$. In the following examples we will choose x, y, u, w so that $\mathscr{V}$ has exactly 10 points, with 9 inside $\bar{\mathbb{D}}$. We then compute the cubature rule $\tilde{v} \equiv \sum_{i=1}^{10} \rho_i \delta_{z_i}$ as in Section 1. The examples were carried out using *Mathematica* [27], with Precision $\to$ 1000, but the results for $x, y, u, w, q_0, q_1, q_2, z_1, \ldots, z_{10}, \rho_1, \ldots, \rho_{10}$ are presented here only to 20 places. We note that system (4.3) is highly sensitive to small changes in the precision and initial values of *Mathematica*'s FindRoot numerical method that we use to solve it. Moreover, replacing (4.3) by a mathematically equivalent system, or replacing one version of the software by another, may lead to a different solution, or even the failure to find a solution.

Example 4.1. System (4.3) is too difficult to solve algebraically. So we use *Mathematica*'s FindRoot method to locate a particular solution. Since we have 8 variables, FindRoot requires a system of 8 equations and initial values for the variables. Thus, we must augment (4.3) with auxiliary equations that will facilitate the search for a solution. The 2 additional equations that we use are simple linear constraints in 2 or 3 of the 8 variables. These constraints are chosen somewhat arbitrarily, on the basis of a small preliminary simulation, with starting points chosen randomly in $[-1, 1]^8$, which yields rules with at least 6 or 7 inside points. Then a finer simulation is run to discover the rule that we present here.

In this example, we augment (4.3) with $\text{test}\,7 = y_1 + w_2/5$ and $\text{test}\,8 = x_2 - w_1/5$. We now solve the augmented system numerically, with the indicated initial values for the variables, using:

```
FindRoot[{test1= =0,test2= =0,test3= =0,test4= =0,test5= =0,test6
= =0,test7= =0,test8= =0}, {x1,0.002}, {x2,-0.0025}, {y1,-0.04}, {y2,
-0.001},{u1,0.001},{u2,0}, {w1,0.55},{w2,0.56}, MaxIterations -> 1000,
WorkingPrecision -> 1000]
```

which leads to the following values for the new moments (rounded here to 20 places):

$$x_1 \approx 0.05875842649995014043, \quad x_2 \approx 0.08690295780758819217,$$
$$y_1 \approx -0.10231469021675854329, \quad y_2 \approx -0.05070631220605340142,$$
$$u_1 \approx 0.11684390655193366350, \quad u_2 \approx 0.01560973819518245391,$$
$$w_1 \approx 0.43451478903794096084, \quad w_2 \approx 0.51157345108379271644.$$

A calculation now shows that the leftmost 3 columns of $W(=M(3)^{-1}B(4))$ are of the following form (displayed here only to 5 places merely to indicate the location of the nonzero entries):

0.	0.	−0.16666
0.78163 + 0.38737i	−0.44888 − 0.66389i	−0.44888 + 0.66389i
−0.89262 − 0.11925i	0.78163 + 0.38737i	−0.44888 − 0.66389i
0.	0.75	0.
0.	0.	1.
0.	0.	0.
0.07481 + 0.11065i	0.07481 − 0.11065i	−0.13027 + 0.06456i
−1.17244 − 0.58105i	0.67332 + 0.99583i	0.67332 − 0.99583i
1.33893 + 0.17887i	−1.17244 − 0.58105i	0.67332 + 0.99583i
0.55324 + 0.65136i	0.14877 + 0.01987i	−0.13027 − 0.06456i.

For $i = 0, 1, 2$, column i of W gives the successive coefficients of $p_i(z, \bar{z}) \equiv \bar{z}^i z^{4-i} - q_i(z, \bar{z})$. Thus we have

$$q_0 = z^4 - (W_{2,1}z + W_{3,1}\bar{z} + W_{7,1}z^3 + W_{8,1}z^2\bar{z} + W_{9,1}z\bar{z}^2 + W_{10,1}\bar{z}^3), \tag{4.4}$$

$$q_1 = z^3\bar{z} - (W_{2,2}z + W_{3,2}\bar{z} + W_{4,2}z^2 + W_{7,2}z^3 + W_{8,2}z^2\bar{z} + W_{9,2}z\bar{z}^2 + W_{10,2}\bar{z}^3), \tag{4.5}$$

$$q_2 = z^2\bar{z}^2 - (W_{1,3} + W_{2,3}z + W_{3,3}\bar{z} + W_{5,3}z\bar{z} + W_{7,3}z^3 + W_{8,3}z^2\bar{z} + W_{9,3}z\bar{z}^2 + W_{10,3}\bar{z}^3). \tag{4.6}$$

Using *Mathematica*'s Solve command, we find that q_0, q_1, q_2 have exactly 10 common zeros (as guaranteed by [10, Theorem 1.2]), 9 of which are inside the unit disk, as follows:

$$z_1 \approx 1.08935297526669100561 + 2.57656275669047696617i,$$
$$z_2 \approx 0.65839515339745152952 - 0.71379406590902948415i,$$
$$z_3 \approx -0.26344579410664713655 + 0.79216632904090436343i,$$
$$z_4 \approx -0.81002547321180270618 + 0.29247575734288437009i,$$
$$z_5 \approx -0.08220761757236031612 7 - 0.79733705482895318777i,$$
$$z_6 \approx 0.91695641111506567059 + 0.10529700092460600829i,$$
$$z_7 \approx 0.47534724539889207225 - 0.25007797877237395465i,$$
$$z_8 \approx 0.43474160333184598201 + 0.61883734600760571921i,$$
$$z_9 \approx -0.20260457859676702432 + 0.03670079845975203033 7i,$$
$$z_{10} \approx -0.72023888089934836719 - 0.44786524025442000232i.$$

The corresponding densities are found using the matrix V (as described near the end of Section 1):

$$\begin{aligned} &\rho_1 \approx 0.00009308114588497672, && \rho_2 \approx 0.12899555049078195312,\\ &\rho_3 \approx 0.31664762531344189537, && \rho_4 \approx 0.27240717016600362994,\\ &\rho_5 \approx 0.37225061581578864761, && \rho_6 \approx 0.18161229215837170183,\\ &\rho_7 \approx 0.49451385953402823059, && \rho_8 \approx 0.43760530571338558483,\\ &\rho_9 \approx 0.64332160342564142566, && \rho_{10} \approx 0.29414554982646519280. \end{aligned}$$

Notice that the density corresponding to z_1, the node outside the disk, is quite small relative to the other densities.

Example 4.2. We augment system (4.3) with the auxiliary tests test $7: y_1 + w_2/2$ and test $8: x_2 - w_1/2$, and then apply

```
FindRoot[{test1= =0,test2= =0,test3= =0,test4= =0,test5= =0,test6
= =0,test7= =0,test8= =0},{x1,0.002},{x2,-0.0025}, {y1,-0.04}, {y2,
-0.001}, {u1,0.001},{u2,0},{w1,0.55}, {w2,0.56},MaxIterations->1000,
WorkingPrecision -> 1000],
```

which yields the following moments of degree 7 for a flat extension $M(4)$:

$$\begin{aligned} &x_1 \approx 0.18647826670148609679, && x_2 \approx 0.25050664264707280813,\\ &y_1 \approx -0.26740382893064394581, && y_2 \approx 0.16750421907152444442,\\ &u_1 \approx -0.16685707807610045987, && u_2 \approx -0.26940290110276574134,\\ &w_1 \approx 0.50101328529414561626, && w_2 \approx 0.53480765786128789161. \end{aligned}$$

A calculation of $W := M(3)^{-1}B(4)$ shows that q_0, q_1, q_2 have the same monomials with nonzero coefficients as in Example 4.1, i.e., they are of the form (4.4)–(4.6). Proceeding as above, we find 10 common zeros (9 inside the disk), and corresponding densities, as follows:

$$\begin{aligned} &z_1 \approx 5.15176024657790622703 + 6.11746492953987926850\mathrm{i},\\ &\quad \rho_1 \approx 1.52367134918212737 87 \times 10^{-7},\\ &z_2 \approx -0.35194927140783762331 + 0.86268588770248855893\mathrm{i},\\ &\quad \rho_2 \approx 0.17083805820315898856,\\ &z_3 \approx 0.64902147753192716812 - 0.56707758589065668387\mathrm{i},\\ &\quad \rho_3 \approx 0.27133932383688146303,\\ &z_4 \approx -0.91823061374684612406 + 0.23129090837261377945\mathrm{i},\\ &\quad \rho_4 \approx 0.15340031188915924687,\\ &z_5 \approx -0.04231617423905305087 - 0.84307070756503077565\mathrm{i},\\ &\quad \rho_5 \approx 0.30088248415532709158, \end{aligned}$$

$$z_6 \approx -0.40739499425827805737 + 0.32262259100094997320\mathrm{i},$$
$$\rho_6 \approx 0.49330918052475339361,$$
$$z_7 \approx 0.82522378836290927016 + 0.15428832079228612784\mathrm{i},$$
$$\rho_7 \approx 0.30867790532204691920,$$
$$z_8 \approx 0.34663245842639227325 + 0.69464522663849494920\mathrm{i},$$
$$\rho_8 \approx 0.41090185901380751928,$$
$$z_9 \approx 0.14683235480899141164 - 0.14410774068256471240\mathrm{i},$$
$$\rho_9 \approx 0.64348313523902047914,$$
$$z_{10} \approx -0.65094920409606338542 - 0.44964255716659905591\mathrm{i},$$
$$\rho_{10} \approx 0.38876024303850321897.$$

As in the previous example, the density corresponding to the single "outside" point is relatively small.

Example 4.3. We augment system (4.3) with the auxiliary tests test 7 : $10w_1 - u_1$ and test 8 : $10u_2 + x_1 + w_1$, and then apply

```
FindRoot[{test1= =0,test2= =0,test3= =0,test4= =0,test5= =0,test6
= =0,test7= =0,test8= =0},{x1, 0.1},{x2, -0.1},{y1, 0.1}, {y2, 0.0},
{u1,0.1},{u2,0.1},{w1,0},{w2,0},MaxIterations->1000,WorkingPrecision
-> 1000],
WorkingPrecision -> 1000],
```

which yields the following moments of degree 7 for a flat extension $M(4)$:

$$x_1 \approx 0.25320904387463164897, \quad x_2 \approx -0.03643595235676003296,$$
$$y_1 \approx 0.25600215978951379421, \quad y_2 \approx -0.04404590648840138369,$$
$$u_1 \approx 0.25991220885139583393, \quad u_2 \approx -0.02792002647597712324,$$
$$w_1 \approx 0.02599122088513958339, \quad w_2 \approx -0.71012159236778570675.$$

A calculation shows that q_0, q_1, q_2 have the same form as in (4.4)–(4.6), with 10 common zeros (9 inside the disk), and corresponding densities, as follows:

$$z_1 \approx 6.55358190036050584215 - 0.45461134734919122880\mathrm{i},$$
$$\rho_1 \approx 4.99787674016841903155 \times 10^{-7},$$
$$z_2 \approx 0.37746016085359607967 - 0.85047970099245546502\mathrm{i},$$
$$\rho_2 \approx 0.17233042535635923651,$$
$$z_3 \approx -0.63774728185466006389 + 0.55375797790346956403,$$
$$\rho_3 \approx 0.30007237744857386176,$$
$$z_4 \approx 0.04053614224615403855 4 + 0.86086515965431581674,$$
$$\rho_4 \approx 0.27140951482808930078,$$

$$z_5 \approx 0.73902762187825758550 - 0.23128713932410755117\mathrm{i},$$
$$\rho_5 \approx 0.41368199712240079322,$$
$$z_6 \approx -0.05503761662212438952 5 - 0.51789629817736177821\mathrm{i},$$
$$\rho_6 \approx 0.49336826466067161535,$$
$$z_7 \approx -0.77661172655481571874 - 0.15785899090034694779\mathrm{i},$$
$$\rho_7 \approx 0.38666536428001292860,$$
$$z_8 \approx -0.47092520388169095707 - 0.82402758522954932607\mathrm{i},$$
$$\rho_8 \approx 0.15102457933153305051,$$
$$z_9 \approx -0.00483448282953645454 58 + 0.20568997559253167978\mathrm{i},$$
$$\rho_9 \approx 0.64347148861000654824,$$
$$z_{10} \approx 0.68246584131955073295 + 0.48801404100843079075\mathrm{i},$$
$$\rho_{10} \approx 0.30956814216447188665.$$

5. Degree 6 minimal rules for T_2 with 9 nodes inside

Let T_2 denote the triangle bounded by the positive x and y axes and by $y = 1 - x$, and let μ_{T_2} denote planar Lebesgue measure restricted to T_2. Since $M(3)[\mu_{T_2}]$ is invertible and has flat extensions, a minimal rule for μ_{T_2} of degree 6 has 10 nodes, but the size of a minimal inside rule of degree 6 is unknown (cf. [3,15]). In [21], Rasputin proved the existence of a 10-node, degree 6 rule for μ_{T_2} with 9 nodes inside T_2, and this rule was subsequently computed in [13]. We conclude by presenting 3 new rules of this type.

Example 5.1. We follow exactly the same method as in Section 4. So we will omit certain details. A minimal rule of degree 6 corresponds to a flat extension of $M(3)[\mu_{T_2}]$, which depends on a choice of new moments of degree 7, $\gamma_{3,4} \equiv x_1 + ix_2$, $\gamma_{2,5} \equiv y_1 + iy_2$, $\gamma_{1,6} \equiv u_1 + iu_2$, $\gamma_{0,7} \equiv w_1 + iw_2$, such that a system of 6 tests (as in (3.1) and (3.2) and (4.2) and (4.3)) is satisfied. We introduce two auxiliary tests, test7 $= x_2 - 2/105$ and test8 $= y_2 + 1/189$, and use

```
FindRoot[{test1==0,test2==0,test3==0,test4==0,test5==0,test6
==0,test7 == 0,test8 == 0},{x1,2/105},{x2,2/106},{y1,1/189},{y2,-1/
190.},{u1, 1/63},{u2, 1/64.1},{w1,0.0000},{w2,0}, MaxIterations -> 1000,
WorkingPrecision -> 1000]
```

to determine the following solution:

$$x_1 \approx 0.01903690131417828357, \qquad x_2 \approx 0.01904761904761904762,$$
$$y_1 \approx 0.00532676231323564172, \qquad y_2 \approx -0.00529100529100529101,$$
$$u_1 \approx 0.01585077013177522147, \qquad u_2 \approx 0.01582266313859144975,$$
$$w_1 \approx -0.00004270632646608794, \qquad w_2 \approx 0.00002175914840384059.$$

Proceeding exactly as in Section 4, we find that the atoms and densities corresponding to this flat extension are as follows:

$$
\begin{aligned}
z_1 &\approx 0.05451282182629403916 + 0.86403677035875354383\mathrm{i},\\
\rho_1 &\approx 0.02887267769434291170;\\
z_2 &\approx 0.86828451109541758480 + 0.06202916939015514617\mathrm{i},\\
\rho_2 &\approx 0.02791150626727831388;\\
z_3 &\approx 0.06858222550200364334 + 0.54207861088382750537\mathrm{i},\\
\rho_3 &\approx 0.06552324520720880444;\\
z_4 &\approx 0.58760657399475662285 + 0.06919744706636750934\mathrm{i},\\
\rho_4 &\approx 0.05987158344101040305;\\
z_5 &\approx 0.28426028659961631648 + 0.63294016318550047571\mathrm{i},\\
\rho_5 &\approx 0.06291163183794948397;\\
z_6 &\approx -0.13752010210756671877 - 0.30788256423433436770\mathrm{i},\\
\rho_6 &\approx 0.00010393156989535370;\\
z_7 &\approx 0.61223728334283590580 + 0.30880911134803945694\mathrm{i},\\
\rho_7 &\approx 0.06253879130134126597;\\
z_8 &\approx 0.23578370624558595124 + 0.04511819908763519725\mathrm{i},\\
\rho_8 &\approx 0.04121064930493564169;\\
z_9 &\approx 0.05287131102766968847 + 0.16435946557161212930\mathrm{i},\\
\rho_9 &\approx 0.04583019225024005898;\\
z_{10} &\approx 0.31252293869846869423 + 0.30253751067628073277\mathrm{i},\\
\rho_{10} &\approx 0.10522579112579768661.
\end{aligned}
$$

Thus 9 atoms are inside T_2, and the weight for the outside point is relatively small.

The next example is based on a change of variables argument. We thank Professor Raúl Curto for valuable discussions concerning this approach.

Example 5.2. We begin by generating a 10-node (minimal) rule of degree 6 for $\mu \equiv \mu_{T'}$, planar measure on the equilateral triangle T' with vertices $(-1, 0)$, $(1/2, \sqrt{3}/2)$, $(1/2, -\sqrt{3}/2)$. For this, we compute a choice of new moments of degree 7, $\gamma_{3,4} \equiv x_1 + ix_2$, $\gamma_{2,5} \equiv y_1 + iy_2$, $\gamma_{1,6} \equiv u_1 + iu_2$, $\gamma_{0,7} \equiv w_1 + iw_2$, which satisfy the 6 tests required for a flat extension of $M(3)[\mu_{T'}]$, as well as the auxiliary tests test7 $= x_1 + 100u_1$ and test8 $= y_1 - 600w_1$. From a calculation with

```
FindRoot[ {test1==0, test2==0, test3==0,test4==0, test5==0,
test6==0, test7==0, test8==0}, {x1,-0.747502}, {x2,0.461809}, {y1,
0.818657},{y2, -0.567856},{u1, 0.399292},{u2, 0.13118},{w1, 0.376431},
{w2, 0.845715},
WorkingPrecision -> 1000, MaxIterations -> 1000]
```

we obtain

$$
\begin{aligned}
x_1 &\approx 0.00219762792350137480, & x_2 &\approx 0.00367385304850666266,\\
y_1 &\approx -0.04989866768738982723, & y_2 &\approx 0.00003791833712168485,\\
u_1 &\approx -0.00002197627923501375, & u_2 &\approx -0.00014639829698948524,\\
w_1 &\approx -0.00008316444614564971, & w_2 &\approx -0.00015448127947472979.
\end{aligned}
$$

Using the same method as in the previous examples, we construct a 10-node, degree 6 cubature rule R for $\mu_{T'}$, with points (x_i, y_i), $(1 \leqslant i \leqslant 10)$, 9 of which are inside T'. We now consider the mapping $\mathscr{T} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathscr{T}(x, y) = ((1+x)/3 - y/\sqrt{3}, (1+x)/3 + y/\sqrt{3})$. $\mathscr{T}$ is an injective, differentiable mapping of the plane onto itself which carries $X \equiv \operatorname{int} T'$ onto $Y \equiv \operatorname{int} T_2$. Since $\mathscr{T}$ is degree-one, it preserves the degrees of polynomials. It follows from the change of variables theorem [23, Theorem 7.26, p. 154] that $\mathscr{T}$ transforms the nodes of any cubature rule $R_{T'}$ for $\mu_{T'}$ into the nodes of a cubature rule R_{T_2} (of the same degree) for μ_{T_2}. Under this transformation, each weight w of $R_{T'}$ corresponds to a weight $|J|w$ in R_{T_2}, where J is the Jacobian of T.

Applying the preceding to $R_{T'} \equiv R$, we obtain a 10-node rule of degree 6 for T_2, with 9 points inside, as follows:

$$
\begin{aligned}
z_0 &\approx 0.86667628720079301151 + 0.07565405977346918862\mathrm{i},\\
&\rho_0 \approx 0.02826400957658927345;\\
z_1 &\approx -0.16959879307066449318 + 1.34523523125896786049\mathrm{i},\\
&\rho_1 \approx 0.00021370477251372711;\\
z_2 &\approx 0.05927026719750294040 + 0.07617418467670494201\mathrm{i},\\
&\rho_2 \approx 0.02909817601047829277;\\
z_3 &\approx 0.06890478116563815633 + 0.36813220156515319208\mathrm{i},\\
&\rho_3 \approx 0.06288800576793462391;\\
z_4 &\approx 0.62498728609879507225 + 0.08046360474525314117\mathrm{i},\\
&\rho_4 \approx 0.06222407211501817460;\\
z_5 &\approx 0.56487264160193274429 + 0.36628298135039822008\mathrm{i},\\
&\rho_5 \approx 0.06259128786741222565;\\
z_6 &\approx 0.04829035703162200822 + 0.75523777384946982045\mathrm{i},\\
&\rho_6 \approx 0.04351501958940286419;\\
z_7 &\approx 0.29905130978726442711 + 0.08067235417378553980\mathrm{i},\\
&\rho_7 \approx 0.06271506654532347954;\\
z_8 &\approx 0.19942329544123185279 + 0.75260530151089574366\mathrm{i},\\
&\rho_8 \approx 0.04330246574148926867;\\
z_9 &\approx 0.30813310095840269271 + 0.38433159886604123853\mathrm{i},\\
&\rho_9 \approx 0.10518819201383807011.
\end{aligned}
$$

Example 5.3. As in the last example, we first generate a minimal rule for the equilateral triangle T'. We augment the 6 basic tests with test7 $= x_1 + 200z_1$ and test8 $= y_1 - 600w_1$, and then employ

```
FindRoot[{test1 = =0, test2 =  =0, test3 = =0,test4= =0, test5= =0,
test6= =0, test7= =0, test8= =0}, {x1,0.0968672}, {x2,0.216784}, {y1,
0.58715}, {y2, 0.0469904}, {u1,-0.743291}, {u2,0.184899}, {w1,0.931683},
{w2, .204699},
WorkingPrecision -> 1000, MaxIterations -> 1000]
```

to determine new moments of degree 7, as follows:

$$x_1 \approx 0.002189450867044244889, \quad x_2 \approx -0.00367844169813341585,$$
$$y_1 \approx -0.04989859803234735600, \quad y_2 \approx -0.000033682664549027 35,$$
$$u_1 \approx -0.0000109472543352 2122, \quad u_2 \approx 0.00014012486136942483,$$
$$w_1 \approx -0.000083164330053912 26, \quad w_2 \approx 0.00015435464605161459.$$

The resulting flat extension corresponds to a 10-node cubature rule of degree 6 for $\mu_{T'}$ with 9 points inside T'. Applying the transformation $\mathscr{T}$ exactly as in Example 5.2, we derive a 10-node degree 6 rule for μ_{T_2} with 9 points inside T_2, as follows:

$$z_0 \approx 1.34520449171804650248 - 0.16997705976184266637\mathrm{i},$$
$$\rho_0 \approx 0.02826400957658927345;$$
$$z_1 \approx 0.75277842319456433290 + 0.19922975273245901937\mathrm{i},$$
$$\rho_1 \approx 0.00021370477251372711;$$
$$z_2 \approx 0.75506776295088341843 + 0.04826897198418157603\mathrm{i},$$
$$\rho_2 \approx 0.02909817601047829277;$$
$$z_3 \approx 0.07614022599082027774 + 0.05915252238630340150\mathrm{i},$$
$$\rho_3 \approx 0.06288800576793462391;$$
$$z_4 \approx 0.08065864722486662765 + 0.29871229667234513994\mathrm{i},$$
$$\rho_4 \approx 0.06222407211501817460;$$
$$z_5 \approx 0.08047813886498974549 + 0.62462708714195231397\mathrm{i},$$
$$\rho_5 \approx 0.06259128786741222565;$$
$$z_6 \approx 0.07568892982295682484 + 0.86651653552085023189\mathrm{i},$$
$$\rho_6 \approx 0.04351501958940286419;$$
$$z_7 \approx 0.36640416325052823730 + 0.56474673404854647886\mathrm{i},$$
$$\rho_7 \approx 0.06271506654532347954;$$
$$z_8 \approx 0.38433182066738832021 + 0.30809220504917274799\mathrm{i},$$
$$\rho_8 \approx 0.04330246574148926867;$$
$$z_9 \approx 0.36801210317940869136 + 0.06890012469616222836\mathrm{i},$$
$$\rho_9 \approx 0.10518819201383807011.$$

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